Mathematical Enrichment   Jan 27th, 2018

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Example   IMO (1990/3, Beijing)

Determine all integers $n > 1$ such that

$$\frac{2^n+1}{n^2}$$

is an integer.

Integers: whole numbers, negative, positive or 0

$\ldots, -2, -1, 0, 1, 2, 3, \ldots$

"Natural Numbers" = positive integers

Rational numbers $57/43$, $3/2$, ...

$$3.14 = \frac{314}{1000}$$

"Number Theory"

Diophantine equations ← solving equations only using integers or rational numbers.

Eg. $x^2+y^2=z^2$

Questions involving divisibility and prime numbers.

<u>Recall</u> The integer d divides the integer n if $\underline{n = d \times \text{an integer}}$

We write "$d \mid n$"

$3 \mid 15$ $\quad 3 \nmid 17$

$15 \mid 0$ since $0 = 15 \times 0$

$0 \nmid 2$ since $2 \neq 0 \times$ integer

$0 \mid 0$ since $0 = 0 \times 0$

"d is a divisor or factor of n"

= "n is a multiple of d"

A positive integer $\underline{n > 1}$ is a <u>prime number</u> if the only positive integer divisors are 1 and n.

If $n > 1$ is not prime, we say it is <u>composite</u>.

The prime numbers

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, \ldots$$

<u>Thm</u>

Every integer > 1 always has a prime factor.

In fact, every number > 1 can be written – in essentially one way – as a product of prime numbers.

$$30 = 2 \cdot 3 \cdot 5$$
$$12 = 2 \cdot 2 \cdot 3$$
$$1500 = 2 \cdot 2 \cdot 3 \cdot 5 \cdot 5 \cdot 5$$

"The Fundamental Theorem of Arithmetic".

Is 163 a prime number?

If not prime, it must have a prime factor $\leq \sqrt{163}$

We need only check

$\cancel{2}, \cancel{3}, \cancel{5}, \cancel{7}, \cancel{11}$: It <u>is</u> prime.

Does there a exist a sequence of 1000 consecutive integers none of which is prime?

Consider

$$N = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots \cdots \cdot 1001$$

($N = 1001!$ "1001 factorial")

Now take

$$N+2, \quad N+3, \quad N+4, \quad \ldots, \quad N+1001$$

div by 2   div by 3   div by 4   - -   div 1001.

By this argument, there are arbitrarily long gaps between one prime and the next.

Does the sequence of primes go on forever? How can we be sure?

<u>Answer:</u> Yes (Euclid, 350 BC):

Let $p_1, p_2, \ldots, p_n$ be <u>any</u> finite list of primes. We'll show there is a prime not in this list.

Note that it's enough to show that there is a number not divisible by any of the primes listed.
(Since any such number has <u>some</u> prime factor.)

Solution: Take $N = P_1 \cdot P_2 \cdots P_n + 1$:
this leaves remainder 1 on division by any of $P_1, \dots, P_n$.

<u>primes</u> 2

$4n+3$: 3, 7, 11, 19, ...

$4n+1$: 5, 13, 17, 29, . . .

Is the sequence of primes of the form $4n+3$ infinite?

Let $P_1, \dots, P_m$ be any finite list of primes of the form $4n+3$:

Let $N = 4P_1 \cdots P_m + 3$ (~~p~~leave P_i out if $P_i = 3$)

N is not divisible by $P_1, \dots, P_m$

N ~~is~~ does have prime factors, however.

But note that when we multiply any two numbers of the form $4n+1$ we get another number of the same form:

$$(4n+1)\cdot(4k+1)$$
$$= 16nk + 4k + 4n + 1$$
$$= 4\cdot(4nk+n+k) + 1$$
$$= 4t + 1.$$

So if a, b, c .. are all of the form $4n+1$, so is abc..

but N is <u>not</u> of the form $4n+1$.

So at <u>least one</u> of its prime factors is of the form $4n+3$.

<u>Done</u>

<u>Much harder</u>: To show inf many primes of the form $4n+1$.

Problems

(1) Show that there is an arithmetic progression of integers

$$a,\ a+d,\ a+2d,\ a+3d, \ldots$$

no term of which is prime <u>or</u> a perfect <u>power (square, cube, 4th power ...)</u>

(2) The 1st 5 primes are 2, 3, 5, 7, 11

Can one find 14 consecutive integers ~~none of which is divisible by any of these primes?~~ with the property that each is divisible by at least one of these primes?