The Integer Part Party

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For any real number $x \in \mathbb{R}$ we denote[1] by $[x]$ the *integer part* of x and by $[\![x]\!]$ the *fractional part* of x. That is,

$$[x] = \max\{n \in \mathbb{Z} \mid n \leq x\} \text{ and } x = [x] + [\![x]\!].$$

Note that $[x]$ is always an integer and $[\![x]\!]$ is a real number such that $0 \leq [\![x]\!] < 1$. Observe also, that

$$x - 1 < [x] \leq x < [x] + 1.$$

Warm-up problems

Problem 1. *Find integer and fractional parts of the following numbers:*

$$14, \quad 4.1, \quad -2.6, \quad 35/4, \quad -22/3, \quad \sqrt{14}, \quad \pi, \quad \pi/2 \quad \pi/3$$

Problem 2. *Draw graphs of* $f(x) = [x]$ *and* $g(x) = [\![x]\!]$ *for* $|x| \leq 5$.

Problem 3. *Compute* $[1] + [\sqrt{2}] + [\sqrt{3}] + \ldots + [\sqrt{100}]$.

Problem 4. *Find all* $x \in \mathbb{R}$ *such that* $1 + [x] + [x^2] + [x^3] = [\![x]\!]$.

Problem 5. *Find all* $x \in \mathbb{R}$ *such that* $3[x] = 5[\![x]\!] + 4$.

Problem 6. *Find all* $x \in \mathbb{R}$ *such that* $x^2 + [\![x]\!]^2 = 11$.

Take out the integer trick

Problem 7. *Prove that* $[x + n] = [x] + n$ *for all integers* $n \in \mathbb{Z}$.

Problem 8. *Find all* $x \in \mathbb{R}$ *such that* $[2x + 8] = 8x + 2$.

Problem 9. *Find all* $x \in \mathbb{R}$ *such that* $\left[\frac{2x+1}{3}\right] + \left[\frac{4x+5}{6}\right] = \frac{3x-1}{2}$.

Problem 10. *Find the smallest positive* $x \in \mathbb{R}$ *such that* $[x^2] - x[x] = 2019$.

Problem 11. *Find all integers* $n \in \mathbb{Z}$ *such that*

$$\left[\frac{n}{1!}\right] + \left[\frac{n}{2!}\right] + \left[\frac{n}{3!}\right] = 224.$$

Then find all $x \in \mathbb{R}$ *such that*

$$\left[\frac{x}{1!}\right] + \left[\frac{x}{2!}\right] + \left[\frac{x}{3!}\right] = 224 + [\![x]\!].$$

[1] Notation $[x]$ was introduced by Gauss. In literature, notation $\lfloor x \rfloor$ (floor) is sometimes used instead of $[x]$, and $\lceil x \rceil$ (ceiling) is used instead of $[x] + 1$. Also, $\{x\}$ is sometimes used for $[\![x]\!]$.

De Polignac's formula[2] and trailing zeroes

Suppose $n \in \mathbb{N}$ is a positive integer and p a prime. The largest exponent r such that p^r divides $n!$ is equal to

$$r = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \ldots = \sum_{k=1}^{\infty} \left[\frac{n}{p^k}\right].$$

Observe that there are only finitely many non-zero terms in this sum!

Problem 12. *Use the formula to find the largest exponent of* 3 *such that* 3^r *divides* $3000!$. *(Hint: Use* $[n/(ab)] = [[n/a]/b]$ *for faster computation of terms* $[3000/3^k]$.*)*

Problem 13. *Observe that* $[n/k]$ *is the number of integers between* 1 *and* n *divisible by* k. *Use this to prove De Polignac's formula.*

Problem 14. *Write down* $50!$ *as a product of powers of primes.*

Problem 15. *Find the number* t *of trailing zeroes of* $2020!$*, that is, the number of consecutive zero digits at the end of this number written in decimal notation.*

Problem 16. *Write down the last 50 digits of number* $210! - 1$.

Problem 17. *Find all positive integers* n *such that* $n!$ *has exactly 100 trailing zeroes.*

Miscelaneous problems

Problem 18. *Find all* $n \in \mathbb{N}$ *such that* $[\sqrt[3]{1}] + [\sqrt[3]{2}] + [\sqrt[3]{3}] + \ldots + [\sqrt[3]{n}] = 2n$.

Problem 19. *Compute* $f(n) = [1] + [\sqrt{2}] + [\sqrt{3}] + \ldots + [\sqrt{n}]$ *for* $n = 2020$ *and/or derive the general formula for* $n \in \mathbb{N}$. *(Hint: express the final result in terms of* n *and* $N = [\sqrt{n}]$.*)*

Problem 20. *For* $x \in \mathbb{R}$ *and* $a, b \in \mathbb{N}$*, prove*

$$[x/a] = [[x]/a] \quad \text{and} \quad [x/(ab)] = [[x/a]/b].$$

(REMEBER BOTH FORMULAS! VERY USEFUL!)

Problem 21. *Prove that* $[\sqrt{n} + \sqrt{n+1}] = [\sqrt{4n+2}]$ *for all* $n \in \mathbb{N}$. *(Ramanujan's puzzle)*

Problem 22. *Prove that* $\left[\frac{n+1}{2}\right] + \left[\frac{n+2}{4}\right] + \left[\frac{n+4}{8}\right] + \ldots = n$ *for all positive integers* $n \in \mathbb{N}$.

Problem 23. *Prove the following basic properties of* $[\ ]$:

(a) $[x] + [y] \leq [x+y] \leq [x] + [y] + 1$.

(b) $0 \leq [2x] - 2[x] \leq 1$.

(c) $[x] + [x + 1/2] = [2x]$

(d) $[x] + [-x] = 0$ *if* $x \in \mathbb{Z}$ *and* -1 *otherwise.*

[2] Also called Legendre's formula in the literature.

Partial solutions for The Integer Part Party

1. Caution for $x < 0$, for instance, $[-2.6] = -3$ and $[\![-2.6]\!] = 0.4$.

2. Staircase and toothsaw.

3. $3 \cdot 1 + (8-3) \cdot 2 + (15-8) \cdot 3 + (24-15) \cdot 4 + \ldots + (80-63) \cdot 8 + (99-80) \cdot 9 + 10 = 625$.

4. Observe that the left hand side of the equation is integer, so $[\![x]\!] = 0$. so$x = -1$.

5. Observe that $5[\![x]\!]$ is integer, so $[\![x]\!] = a/5$ for $a = 0, 1, \ldots, 4$. Hence $3[x] = a + 4$ is an integer, so $3|a+4$ and hence $a = 2$, $x = 12/5$.

6. Observe $[\![x]\!]^2 < 1$, so $x^2 > 10$. Thus, $[x] = 3$. Denote $r = [\![x]\!]$ and put $x = 3 + r$ into original equation to obtain 2 solutions for r, but only $r \geq 0$, is good, so $x = 3 + r = \frac{3+\sqrt{13}}{2}$.

7. Since $x = [x] + [\![x]\!]$, we have $[x] + n \leq [x] + [\![x]\!] + n = x + n < [x] + n + 1$. As $[x + n]$ is the largest integer not greater that $x + n$, we get $[x + n] = [x] + n$.

8. Observe that $8x \in \mathbb{Z}$, so $x = y/8$ for $y \in \mathbb{Z}$. Rewrite the equation to get $[y/4] + 6 = y$. Now put $y = 4k + r$, where $r = 0, 1, 2, 3$ and $k \in \mathbb{Z}$ (REMEMBER THIS TRICK!) and rewrite equation into $6 = 3k + r$ using that $[r/4] = 0$. Check possible r to get $r = 0$ or 3, compute k, y and finally $x = 1$ or $7/8$.

9. The right side is integer, so $x = (2y + 1)/3$ for $y \in \mathbb{Z}$. Rewrite equation with y and reduce it to obtain $[(4y+5)/9] + [4y/9 + 1/18] + 1 = y$. Since the nominator is 9, put $y = 9k + r$, where $r = 0, 1, \ldots, 8$ (TRICK: if there were two different fraction nominators, we'd take their LCM instead of 9). Now check all possibilities to get 9 solutions: $x = 3/3, 5/3, 7/3, \ldots, 19/3$.

10. Write $x = n + r$ where $n = [x]$. Rewrite and reduce to get $[2nr + r^2] - nr = 2019$. So $nr \in \mathbb{Z}$ and hence also $2nr \in \mathbb{Z}$. Reduce to $[r^2] + nr = 2019$ and so $nr = 2019$. Then $r = 2019/n < 1$ and so $n > 2019$. Finally, $x = 2020 + \frac{2019}{2020}$.

11. Write $n = 6k + r$ and rewrite and reduce using standard tricks to get $n = 135$ as the only solution. For the second part, observe the right hand side to find the unique x.

12. $[3000/3] + [3000/9] + \ldots = [3000/3] + [1000/3] + [333/3] + \ldots = 1000 + 333 + 111 + 37 + 12 + 4 + 1 = 1498$.

13. Let $k = [n/p]$ and write $1, 2, 3, \ldots, p, p+2, \ldots, 2p, 2p+1, \ldots, kp, kp+1, \ldots, n$ to see why $[n/p]$ counts the numbers between 1 and n. But since some of these are also divisible by p^2, p^3 etc., you have to add all these as well to get the exponent.

14. Use the formula for each prime $p < 47$ to obtain the exponents, say $50! = 2^{47} \cdot 3^{22} \cdot \ldots$.

15. Observe that the number of trailing zeroes depends on the exponents of factors 2 and 5. As the later is smaller, the number of trailing zeroes is just that: $[2020/5] + [2020/5^2] + \ldots = 404 + 80 + 16 + 3 = 503$.

16. Think about the trailing zeroes first.

17. After a little trial and error, we obtain $n = 405, 406, \ldots, 409$. To reduce some of the trial end error, recall the geometric series formula $1 + q + q^2 + \ldots = \frac{1}{1-q}$ for $|q| < 1$. Using this, one can estimate

$$t = [n/5] + [n/5^2] + \ldots \leq n/5 + n/5^2 + \ldots = \frac{n}{5}(1 + 1/5 + \ldots) = n/4,$$

so $t \leq n/4$, giving the lower bound $n \geq 400$.

18. $n = 33$.

19. Observe that the sum is $(1 \cdot 3 + 2 \cdot 5 + \ldots + (N-1)(2(N-1)+1)) + [N^2+1] + [N^2+2] + \ldots + N$. Using the formulas for sum of k and sum of k^2 gives the general formula $\frac{N(6n-2N^2-3N+5)}{6}$.

20. Write $x = [x] + [\![x]\!]$ and $[x] = kn + r$ with $k \in \mathbb{Z}$ and $r \in \{0, \ldots, n-1\}$. Then $\left[\frac{[x]}{n}\right] = \left[\frac{kn+r}{n}\right] = \left[k + \frac{r}{n}\right] = k + \left[\frac{r}{n}\right] = k$, since $[r/n] = 0$. Now

$$\left[\frac{x}{n}\right] = \left[\frac{kn + r + [\![x]\!]}{n}\right] = \left[k + \frac{r + [\![x]\!]}{n}\right] = k + \left[\frac{r + [\![x]\!]}{n}\right] = k,$$

since $0 \leq r + [\![x]\!] < r + 1 < n$. The second part now follows from the first one.

21. Hint: Show $\leq$ and $\geq$ separately.

22. Hint: Use induction for $n \geq 3$.

23. (a) $[x+y]$ is integer and less or equal to $x + y = [x] + [y] + [\![x]\!] + [\![y]\!] \geq [x] + [y] \in \mathbb{Z}$. So $[x+y] \geq [x] + [y]$.

    On the other hand, $[x+y] = [[x] + [y] + [\![x]\!] + [\![y]\!]]$ is also integer. It is the max integer less or equal $[x] + [y] + [\![x]\!] + [\![y]\!] < [x] + [y] + 2 \in \mathbb{Z}$, hence $[x+y] \leq [x] + [y] + 1$.

    (b) Use previous with $x = y$ to get $[x] + [x] \leq [x+x] \leq [x] + [x] + 1$, hence $0 \leq [2x] - 2[x] \leq 1$.

References

[1] Irena Majcen, Smelo na Olimp, 303 naloge iz teorije števil, DMFA Slovenije.

[2] www.brilliant.org.

[3] www.cut-the-knot.org.