MATHEMATICAL ENRICHMENT 14/3/15.

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$$A > B \ ?$$
$$\left(2^{100} \quad 100! \right)$$

1.0001^n $\cancel{2^n}$ $\cancel{n!}$

$a > 1$

a^n
V if n is large enough.
$n!$

A·M - G·M inequality

$a_1, \ldots, a_n \geq 0$

$$\frac{a_1 + \cdots + a_n}{n} \geq (a_1 \cdot a_2 \cdots a_n)^{1/n}$$

$$\underset{\text{AM}}{\uparrow} \qquad\qquad \underset{\text{GM.}}{\uparrow}$$

and equality occurs if and only if $a_1 = a_2 = \cdots = a_n$.

$$a_1 + \cdots + a_n \geq n \cdot \sqrt[n]{a_1 \cdots a_n}$$

(equality only if $a_1 = \cdots = a_n$)

$$a_1 \cdots a_n \leq \left(\frac{a_1 + \cdots + a_n}{n} \right)^n$$

Notation

$$a_1 + a_2 + \cdots + a_i + \cdots + a_n = \sum_{i=1}^{n} a_i$$

$$a_1 \cdot a_2 \cdots \cdot a_n = \prod_{i=1}^{n} a_i$$

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(1) Find the minimum value of $\underset{a_1}{\frac{3}{x}} + \underset{a_2}{\frac{4}{y}} + \underset{a_3}{xy}$, $x, y > 0$

When is this minimum attained?

By AM-GM inequality $a_1+a_2+a_3 \geq 3\cdot\sqrt[3]{12}$

Minimum achieved only if $\frac{3}{x} = \frac{4}{y} = xy = \sqrt[3]{12}$

$$\Rightarrow x = \cancel{\frac{\sqrt[3]{12}}{3}},\ y = \cancel{\frac{\sqrt[3]{12}}{4}},\ \cancel{xy = \frac{\sqrt[3]{12}\cdot\sqrt[3]{12}}{12}}$$

$$\boxed{x = \frac{3}{\sqrt[3]{12}}} \quad \boxed{y = \frac{4}{\sqrt[3]{12}}} \Rightarrow xy = \sqrt[3]{12}$$

$$\left(\text{or } x = \sqrt[3]{\frac{9}{4}} = \frac{\sqrt[3]{3^3}}{\sqrt[3]{12}} = \frac{3}{\sqrt[3]{12}}.\right)$$

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(2) Find the minimum of

$$x^2 + \frac{4}{x} \qquad \text{for all } x > 0.$$

$$x^2 + \frac{2}{x} + \frac{2}{x} \geq 3\sqrt[3]{4}$$

equality? $x^2 = \frac{2}{x} = \sqrt[3]{4}$

$$\boxed{\underline{\text{Solve}}:\quad x = \sqrt[3]{2}}$$

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(3) $(a+b)(b+c)(c+a) \geq 8abc$ if $a,b,c \geq 0$

Show

By AM-GM
$$a+b \geq 2\sqrt{ab}$$
$$b+c \geq 2\sqrt{bc}$$
$$c+a \geq 2\sqrt{ac}$$

$$\therefore \text{prod} \geq 2\cdot 2\cdot 2\cdot \sqrt{ab}\cdot\sqrt{bc}\cdot\sqrt{ac}$$

$$\sqrt{ab}\cdot\sqrt{bc}\cdot\sqrt{ac} = \sqrt{a^2b^2c^2} = abc, \qquad 2\cdot 2\cdot 2\cdot\sqrt{ab}\cdot\sqrt{bc}\cdot\sqrt{ac} = 8\cdot abc$$

(4) Show $x^2+y^2+z^2 \geqslant xy+yz+zx$ when $x,y,z \geqslant 0$

$$\frac{x^2+y^2}{2} \geqslant xy$$

$$\frac{x^2+z^2}{2} \geqslant xz$$

$$\frac{y^2+z^2}{2} \geqslant yz$$

add add

Example Show that

$$x^4+y^4+18 \geqslant 12xy \quad \text{if } x,y>0$$

Solution $$x^4+y^4+9+9 \geqslant 4\cdot\sqrt[4]{x^4y^4 9\cdot 9} = 4\cdot x\cdot y\cdot 3 = 12xy.$$

(equality $x=y=\sqrt[4]{9}$).

Exercises Find the minimum of

$$\frac{2}{y}+\frac{3}{x}+x^2y \quad \text{for } x,y>0$$

Use the AM-GM inequality to find the minimum of

$$\frac{1}{x}+\frac{1}{3-x} \quad \text{when} \quad 0<x<3$$

Find the maximum of $x^2(15-x)$ for $x>0$

Find the maximum of $xy(9-x-y)$ for $x, y > 0$. 4

<u>Solution</u> By AM-GM

$$x \cdot y \cdot (9-x-y) \leq \left(\frac{x+y+9-\cancel{x}-y}{3}\right)^3 = 3^3 = 27.$$

equality occurs $x = y = 9-x-y = 3$ ✓

<u>Recall strategy</u> To prove $A \geqslant B$,

show $A - B = X^2$ or $X^2 + Y^2 + Z^2 + \dots$

<u>Typical Example</u> $x, y \geqslant 0$ Show that

$$\underbrace{1 + x^2 + x^2y^2 + y^2 + 4xy}_{A} \geqslant \underbrace{2x + 2xy^2 + 2x^2y + 2y}_{B}$$

<u>Solution</u> $A - B = 1 + x^2 + x^2y^2 + y^2 + 4xy - 2x - 2xy^2 - 2x^2y - 2y$

$$= 1 + x^2 - 2x + \underbrace{4xy - 2x^2y - 2y}_{+2y(2x - x^2 - 1)} + y^2(x^2 + 1 - 2x)$$

$$= (1 + x^2 - 2x) \cdot (1 - 2y + y^2)$$

$$= (1-x)^2 \cdot (1-y)^2 = [(1-x)(1-y)]^2$$

This is 0 precisely when

<u>either</u> $x = 1$ <u>or</u> $y = 1$.

Cauchy - Schwarz inequality

n = 2 $\quad a_1, a_2 \quad b_1, b_1 \quad$ all nonzero.

Then

$$\overbrace{(a_1b_1 + a_2b_2)^2}^{A} \leq \overbrace{(a_1^2 + a_2^2)\cdot(b_1^2 + b_2^2)}^{B}$$

equality occurs if and only if $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = c.$

$$\text{(ie. } (a_1, a_2) = c\cdot(b_1, b_2) \text{)}$$

" (a_1, a_2) is <u>proportional</u> to (b_1, b_2) ".

~~General case~~ (<u>Proof</u>: $B - A = (a_1b_2 - a_2b_1)^2$).

<u>General case</u> $\quad a_1, a_2, .., a_n, \quad b_1, b_2, .., b_n \quad$ [(all nonzero).]

Then

$$(a_1b_1 + a_2b_2 + \cdot\ + a_nb_n)^2 \leq (a_1^2 + .. + a_n^2)\cdot(b_1^2 + .. + b_n^2)$$

equality occurs if and only if

$$(a_1, .., a_n) = c\cdot(b_1, .., b_n) \quad \text{for some } c.$$

<u>Examples</u> Show that if $x_1, .., x_n > 0$

$$\frac{1}{x_1} + \cdots + \frac{1}{x_n} \geq \frac{n^2}{x_1 + x_2 + .. + x_n}$$

(with equality when all x_i are equal.)

<u>Solution</u> $\quad \left(\frac{1}{x_1} + \cdot\cdot + \frac{1}{x_n}\right)\cdot(x_1 + \cdot\cdot + x_n) \geq n^2 \quad \forall x_1, .., x_n \geq 0.$

Take $a_1 = \sqrt{\frac{1}{x_1}}, \ldots a_n = \sqrt{\frac{1}{x_n}}, \; b_1 = \sqrt{x_1}, .., b_n = \sqrt{x_n}$ and $a_1b_1 = a_2b_2 = \cdot$
$= a_nb_n = 1.$

Find the maximum of

$3x+4y+5z$ on the sphere $x^2+y^2+z^2=1$.

<u>Solution</u> By C-S

$$(3x+4y+5z)^2 \leq \underbrace{(3^2+4^2+5^2)}_{=50}\underbrace{(x^2+y^2+z^2)}_{=1}$$

$$\Rightarrow \quad 3x+4y+5z \leq \sqrt{50} = 5\sqrt{2}$$

<u>equality</u>? $(x,y,z) = c\cdot(3,4,5)$

$$1 = x^2+y^2+z^2 = c^2\cdot(3^2+4^2+5^2) = c^2\cdot 50$$

$$\Rightarrow \quad c = \frac{1}{\sqrt{50}}.$$

So $(x,y,z) = \frac{1}{\sqrt{50}}\cdot(3,4,5)$ works.

<u>Ex</u> Find the maximum value of $5x-y+4z$ on the ellipsoid $x^2+4y^2+z^2=3$.
