THE PRINCIPLE OF INDUCTION

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The Principle of Induction: Let a be an integer, and let $P(n)$ be a statement (or proposition) about n for each integer $n \geq a$. The principle of induction is a way of proving that $P(n)$ is true for all integers $n \geq a$. It works in two steps:

(a) [**Base case:**] Prove that $P(a)$ is true.
(b) [**Inductive step:**] Assume that $P(k)$ is true for some integer $k \geq a$, and use this to prove that $P(k+1)$ is true.

Then we may conclude that $P(n)$ is true for all integers $n \geq a$.

This principle is very useful in problem solving, especially when we observe a *pattern* and want to prove it.

The trick to using the Principle of Induction properly is to spot *how to use* $P(k)$ to prove $P(k+1)$. Sometimes this must be done rather ingeniously!

Problem 1. Prove that for any integer $n \geq 1$,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} .$$

Solution. Let $P(n)$ denote the proposition to be proved. First let's examine $P(1)$: this states that

$$1 = \frac{1(2)}{2} = 1$$

which is correct.

Next, we assume that $P(k)$ is true for some positive integer k, i.e.

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2} .$$

and we want to use this to prove $P(k+1)$, i.e.

$$1 + 2 + 3 + \cdots + (k+1) = \frac{(k+1)(k+2)}{2} .$$

Taking the LHS and using $P(k)$,

$$\begin{aligned} 1 + 2 + 3 + \cdots + (k+1) &= (1 + 2 + 3 + \cdots + k) + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

and thus $P(k+1)$ is true. This completes the proof.

Problem 2. Find a formula for the sum of the first n odd numbers.

Solution. Note that this time we are not told the formula that we have to prove; we have to find it ourselves! Let's try some small numbers and see if a pattern emerges:

$$1 = 1; \quad 1+3 = 4; \quad 1+3+5 = 9;$$
$$1+3+5+7 = 16; \quad 1+3+5+7+9 = 25;$$

We conjecture (guess) that the sum of the first n odd numbers is equal to n^2. Now let's prove this proposition using the principle of induction; call it $P(n)$.
Our statement $P(n)$ is that

$$1+3+5+7+\cdots+(2n-1) = n^2 .$$

First we prove the base case $P(1)$, i.e.

$$1 = 1^2$$

This is certainly true. Now we assume that $P(k)$ is true, i.e.

$$1+3+5+7+\cdots+(2k-1) = k^2 .$$

and consider $P(k+1)$:

$$1+3+5+7+\cdots+(2k+1) = (k+1)^2 .$$

Taking the LHS and using $P(k)$,

$$\begin{aligned} 1+3+5+\cdots+(2k+1) &= (1+3+5+\cdots+(2k-1))+(2k+1) \\ &= k^2+(2k+1) \\ &= (k+1)^2\ . \end{aligned}$$

and thus $P(k+1)$ is true. This completes the proof.

Remark This result can also be proved by dividing a square into L-shaped regions.

Problem 3: Finding Triangles $2n$ points are given in space, where $n \geq 2$. Altogether $n^2 + 1$ line segments ('edges') are drawn between these points. Show that there is at least one set of three points which are joined pairwise by line segments (i.e. show that there exists a *triangle*).

Solution. We will first argue that the proposition (let's call it $P(n)$) holds for $n = 2$.

In the case $n = 2$, there are $2n = 4$ points in space and $n^2 + 1 = 5$ edges.

There are only 6 possible edges connecting 4 points, so in our configuration of 5 edges, one of the 6 must be missing. Suppose the missing edge connects points A and B. Denote the other two points by C and D.

Then there is a triangle ACD (and indeed another one BCD). This proves $P(2)$.

Now let us suppose that the proposition $P(n)$ is true for $n = k$, i.e. that if $2k$ points are joined together by $k^2 + 1$ edges, there must exist a triangle. We seek to prove $P(k+1)$.

In the case of $P(k+1)$ we consider $2(k+1) = 2k+2$ points, which are connected by $(k+1)^2 + 1 = k^2 + 2k + 2$ edges.

Take a pair any pair of points A, B which are joined by an edge .The remaining $2k$ points form a set which we will call $\mathcal{S}$. We count the number of edges in $\mathcal{S}$. There are two possibilities: there are either at least k^2+1 edges in $\mathcal{S}$ or there are at most k^2.

In the first case, we are done. $P(k+1)$ follows, because $P(k)$ implies that $\mathcal{S}$ contains a triangle.

In the second case, we can count the edges as follows:

- There is one edge connecting A to B.
- There are at most k^2 edges connecting pairs of points in $\mathcal{S}$
- As there are k^2+2k+2 edges in total, there must therefore be at least $2k+1$ edges connecting points in S to A or to B.

Finally, we note that as the set $\mathcal{S}$ has $2k$ elements but $2k+1$ edges connecting $\mathcal{S}$ to A or to B, then there must be at least one point, $C \in \mathcal{S}$, connected both to A and to B.

Our triangle is then ABC.

This proves $P(k+1)$.

By induction, then $P(n)$ holds for all integers $n \geq 2$.

Remark. If we have $2n$ points and *exactly* n^2 edges, it is possible to *avoid* making a triangle. This is done by breaking the set of points into two subsets $\mathcal{X}$ and $\mathcal{Y}$ which contain n points each, then connecting every point in $\mathcal{X}$ to every point in $\mathcal{Y}$.

Problem 4: Fibonacci Numbers

The Fibonacci numbers are given by:

$$F_1 = 1$$

$$F_2 = 1$$

$$F_3 = 2$$

$$F_4 = 3$$

$$F_5 = 5$$

$$F_6 = 8$$

$$F_7 = 13$$

$$F_8 = 21$$

$$F_{n+1} = F_{n-1} + F_n$$

Prove that, for $n = 0, 1, 2, \ldots$:

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]$$

Solution. In this case, the inductive proof is more subtle. We cannot show that $P(k)$ implies $P(k+1)$.

We **can** show that $P(k-1)$ and $P(k)$ together imply $P(k+1)$.

To start the induction, we then need to demonstrate $P(n)$ for two consecutive values of n. We do this for $n = 0$ and $n = 1$.

To verify $P(0)$ we have:

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^0 - \left(\frac{1-\sqrt{5}}{2}\right)^0\right] = \frac{1-1}{\sqrt{5}} = 0 = F_0$$

To verify $P(1)$, we have

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1\right] = \frac{\sqrt{5}}{\sqrt{5}} = 1 = F_1$$

Now let us suppose $P(k-1)$ and $P(k)$ both hold, so that:

$$F_{k-1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right]$$

$$F_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right]$$

Then we can derive an expression for F_{k+1} which is:

$$
\begin{aligned}
F_{k+1} &= F_{k-1} + F_k \\
&= \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{k-1} + \left(\frac{1+\sqrt{5}}{2} \right)^{k} \right] \\
&- \frac{1}{\sqrt{5}} \left[\left(\frac{1-\sqrt{5}}{2} \right)^{k-1} + \left(\frac{1-\sqrt{5}}{2} \right)^{k} \right] \\
&= \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{k-1} \left[1 + \frac{1+\sqrt{5}}{2} \right] \\
&- \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^{k-1} \left[1 + \frac{1-\sqrt{5}}{2} \right] \\
&= \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{k+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{k+1} \right]
\end{aligned}
$$

This is the inductive hypothesis we wished to prove.

In the last line, we used the identity:

$$1 + \frac{1 \pm \sqrt{5}}{2} = \left(\frac{1 \pm \sqrt{5}}{2} \right)^2$$

Problem 5: Irrationality of $\sqrt{2}$

Let n be a positive integer. Prove that $\sqrt{2}n$ is not an integer.

Proof: The proposition holds when $n = 1$, because $1 < \sqrt{2} < 2$.

Let us suppose inductively that the proposition holds for $n = 1, 2, 3, \ldots k-1$ for some positive integer $k \geq 2$.

We want to show that $\sqrt{2}k$ is not an integer.

Imagine a number line with the points 0, k, $\sqrt{2}k$ and $2k$ marked on it. As $1 < \sqrt{2} < 2$ then these points are in increasing sequence.

We focus on the intervals from k to $\sqrt{2}k$ and from $\sqrt{2}k$ to $2k$.

If it happens that $\sqrt{2}k$ is an integer, then these lengths are both integers,

But comparing the interval lengths, we note that:

$$\sqrt{2}(\sqrt{2}k - k) = 2k - \sqrt{2}k$$

This cannot happen by proposition $P(\sqrt{2}k - k)$.

Therefore the remaining possibility is that $\sqrt{2}k$ is **not** an integer - which is the proposition $P(k)$ we set out to prove.

This completes the inductive hypothesis. As the proposition holds when $n = 1$, it therefore holds for all positive integers n.

Problems 6 & 7: Quadratic Recursion Let $\{S_n : n = 1, 2, 3, ..\}$ be a sequence of integers, defined by:

$$S_1 = 1$$

$$S_2 = 1$$

$$S_3 = 2$$

$$S_4 = 5$$

$$S_5 = 29$$

$$S_6 = 866$$

$$S_{n+1} = S_{n-1}^2 + S_n^2$$

Show that :

- (Problem 6) If $n = 1, 2, 3, \ldots$ then S_n is not divisible by 7.
- (Problem 7) $S_n \leq 2^{(2^{n-2.7})}$

Solution to Problem 6: Method 1 We look at the remainder of S_n on division by 7. We notice a pattern of period 3 which we then try to prove by induction. Our proposition is as follows:

Proposition $P(n)$: Let n be a positive integer. Then:

- $S_{3n+1} + 2$ is divisible by 7
- $S_{3n+2} - 1$ is divisible by 7
- $S_{3n+3} + 2$ is divisible by 7

We show this in the case $n = 1$ by direct computation:

- $S_4 + 2 = 7$
- $S_5 - 1 = 28 = 4 \times 7$
- $S_6 + 2 = 868 = 124 \times 7$

Thus, $P(1)$ holds.

Let us suppose now that $P(k-1)$ holds for some $k \geq 2$. We can then compute, using the difference of two squares:

$$S_{3k+1} + 2 = (S_{3k} + 2)(S_{3k} - 2) + (S_{3k-1} - 1)(S_{3k-1} + 1) + 7$$

$$S_{3k+2} - 1 = (S_{3k+1} + 2)(S_{3k+1} - 2) + (S_{3k} + 2)(S_{3k} - 2) + 7$$

$$S_{3k+3} + 2 = (S_{3k+2} - 1)(S_{3k+2} + 1) + (S_{3k+1} + 2)(S_{3k+1} - 2) + 7$$

All the terms on the right hand side are multiples of 7, either by $P(k-1)$ or by the previous bullets in the list.

The left hand side is the subject of $P(k)$. Thus, $P(k)$ is proven from $P(k-1)$ and so by induction $P(n)$ holds for all $n = 1, 2, 3, \ldots$.

We are not quite there, as we have only shown that S_n is not a multiple of 7 for $n \geq 4$.

We go back and check S_1, S_2 and S_3 manually.

Finally, then, we have shown that S_n is not a multiple of 7 for any positive n.

Solution to Problem 6: Method 2

We can alternatively prove a simpler inductive hypothesis $P(n)$: S_n not a multiple of 7, with a more tedious inductive step.

Let us suppose we have proved $P(k-1)$ and $P(k)$, so that neither S_{k-1} not S_k are multiples of 7. We can then tabulate S_{k+1} modulo 7 for all combinations of S_{k-1} and S_k modulo 7. The table is as follows:

	1	2	3	4	5	6
1	2	5	3	3	5	2
2	5	1	6	6	1	5
3	3	6	4	4	6	3
4	3	6	4	4	6	3
5	5	1	6	6	1	5
6	2	5	3	3	5	2

As there are no zeros in this table, by checking all the cases we have shown that S_{k+1} is not a multiple of 7.

Remark: If can be shown the corresponding result holds for primes $p=$ 3, 11, 19 and any prime for which $p+1$ is a multiple of 4, but this involves more difficult maths than would be expected for IMO.

Problem 7 Solution Remember the definition of the sequence:

$$S_1 = 1$$

$$S_2 = 1$$

$$S_{n+1} = S_{n-1}^2 + S_n^2$$

We are required to prove the statement $P(n)$ that $S_n \leq 2^{(2^{n-2.7})}$.

Abortive Proof Attempt: Suppose we try to prove this by induction. So let us take the inductive hypothesis for $n = k - 1$ and $n = k$:

$$S_{k-1} \leq 2^{2^{k-3.7}}$$

$$S_k \leq 2^{2^{k-2.7}}$$

We square each side:

$$S_{k-1}^2 \leq 2^{2^{k-2.7}}$$

$$S_k^2 \leq 2^{2^{k-1.7}}$$

Adding these together, we have:

$$S_{k-1}^2 + S_k^2 \leq 2^{2^{k-1.7}} + 2^{2^{k-2.7}}$$

What we **wanted** to prove was the stronger statement $P(k+1)$, that

$$S_{k-1}^2 + S_k^2 \leq 2^{2^{k-1.7}}$$

But unfortunately the inductive step didn't work.

Second Attempt at Induction: We try instead to prove a stronger statement $P'(n)$ which is $S_n \leq 2^{(2^{n-2.7})} - \frac{1}{2}$.

So let us take the new inductive hypothesis for $n = k-1$ and $n = k$:

$$S_{k-1} \leq 2^{2^{k-3.7}} - \frac{1}{2}$$

$$S_k \leq 2^{2^{k-2.7}} - \frac{1}{2}$$

We square each side:

$$S_{k-1}^2 \leq 2^{2^{k-2.7}} - 2^{2^{k-3.7}} + \frac{1}{4}$$

$$S_k^2 \leq 2^{2^{k-1.7}} - 2^{2^{k-2.7}} + \frac{1}{4}$$

Adding these together,

$$S_{k-1}^2 + S_k^2 \leq 2^{2^{k-1.7}} - 2^{2^{k-3.7}} + \frac{1}{2}$$

$$\leq 2^{2^{k-1.7}} - \frac{1}{2}$$

Here, the inductive step works and we have proved $P'(k+1)$.

It remains only to check $P'(n)$ for $n = 1$ and $n = 2$ to complete the inductive statement.

Here we hit another problem, that $P'(n)$ is **false** for some small n. Specifically, we have:

n	S_n	$2^{2^{n-2.7}}$	$P(n)$	$P'(n)$
1	1	1.24	TRUE	FALSE
2	1	1.53	TRUE	TRUE
3	2	2.35	TRUE	FALSE
4	5	5.51	TRUE	TRUE
5	29	30.37	TRUE	TRUE

To put the pieces together, we use the following arguments:

- If $n = 1, 2, 3$, we see that $P(n)$ holds by direct calculation.
- If $n = 4, 5$, we see that $P'(n)$ holds by direct calculation.
- If $n \geq 5$, then $P'(n)$ holds by induction.
- As $P'(n) \implies P(n)$ we conclude that $P(n)$ holds for all positive integers n.

Pólya's Paradox:

A common way (in 1950, at least!) of expressing that something is out of the ordinary is "*That's* a horse of a different color!" The famous mathematician George Pólya gave the following proof that "all horses are the same color", which works by the principle of induction:

Proposition $P(n)$: Suppose we have n horses. Then all n horses are the same colour.

Base case: $n = 1$; if there is only one horse, there is only one colour.

Inductive step: Assume that $P(k)$ is true, i.e. that for any set of k horses, there is only one color. Now look at any set of $k+1$ horses; call this $\{H_1, H_2, H_3, \cdots, H_k, H_{k+1}\}$. Consider the sets $\{H_1, H_2, H_3, \cdots, H_k\}$ and $\{H_2, H_3, H_4, ..., H_{k+1}\}$. Each is a set of only k horses, therefore within each there is only one colour. But the two sets overlap, so there must be only one colour among all $k+1$ horses.

The flaw is that when $k = 2$ the inductive step doesn't work, because the statement that "the two sets overlap" is false.

IMO Problem, 1964. Seventeen scientists correspond with one another. The correspondence is about three topics; any two scientists write to each other about one topic only.

Prove that at least three scientists write to one another on the same topic.