Aspects of Combinatorics

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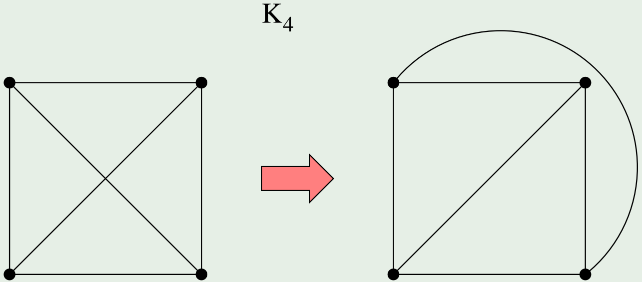
1. Euler's Formula

Definition

We call a graph $G(V, E)$ planar if it can be represented in the plane by points and arcs in such a way that edges meet only at vertices, i.e. they do not cross one another.

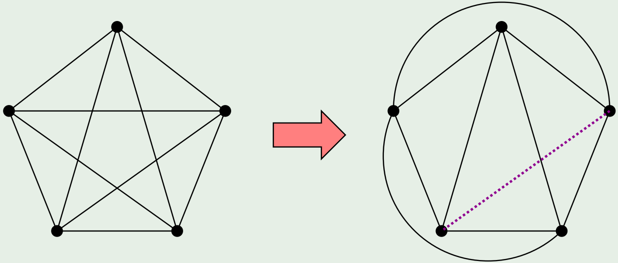
Example

K_4 looks non-planar but is planar.



Example

K_5 is non-planar.

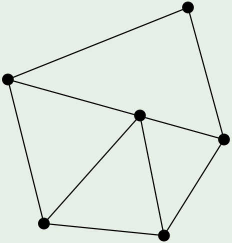


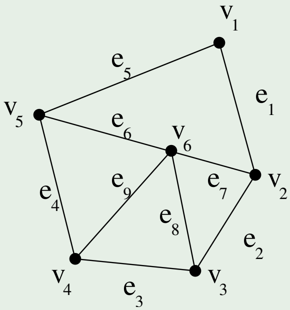
Theorem

Euler Let a (finite non-empty connected) planar graph G have vertex, edge and face sets V, E and F, respectively. Then

$$|V| + |F| = |E| + 1.$$

Example





In this graph, $|V| = 6$, $|E| = 9$, $|F| = 4$, hence

$$|V| + |F| = |E| + 1.$$

Proof of Euler's Formula:

We will prove this result by induction.

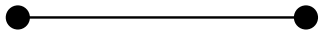
- If there are no edges then the graph is a single vertex.
  $\Rightarrow |V| = 1$, $|F| = 0$, $|E| = 0$ and so the formula

$$|V| + |F| = |E| + 1$$

  holds.
- If there is 1 edge in the graph then it has 2 vertices, but no faces.
  $\Rightarrow |V| = 2$, $|E| = 1$, $|F| = 0$ and the formula

$$|V| + |F| = |E| + 1,$$

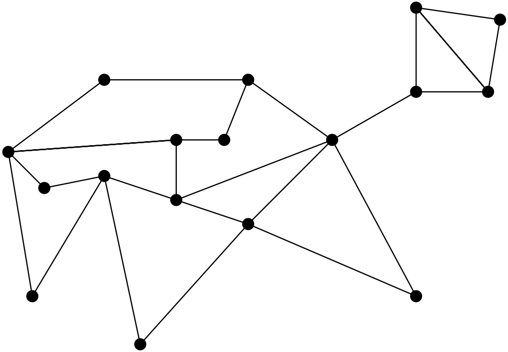
  again, holds.



- Let us assume that Euler's formula is true for all (connected) planar graphs with less than n edges.

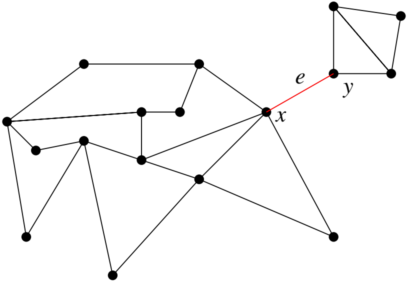
- Consider now a connected planar graph G with vertices V, edges E such that the number of edges $|E| = n$.

  In such a graph there is at least one external edge e, with end vertices x and y, say.

  We now consider removing this edge e and examine the resulting graphs that arise. There are two distinct cases:

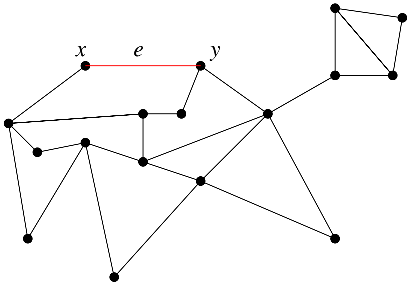
(1) If the removal of the edge e causes the graph to split into two connected graphs G_1 and G_2. Let G_1 have vertex set V_1, edge set E_1 and faces F_1. Similarly, let G_2 have vertex set V_2, edge set E_2 and face set F_2.



Since we have removed an edge from G, both of the graphs G_1 and G_2 have at most $n-1$ edges. From our assumption, we know that $|V_1|+|F_1| = |E_1|+1$ and $|V_2|+|F_2| = |E_2|+1$. Hence

$$
\begin{aligned}
|V|+|F| &= |V_1|+|F_1|+|V_2|+|F_2| \\
&= |E_1|+1+|E_2|+1 \\
&= (|E_1|+1+|E_2|)+1 = |E|+1.
\end{aligned}
$$

(2) If, when we remove the edge e from the planar graph G, we still have a connected planar graph, then call the resulting graph H. Let the vertex/edge and face sets of H be V_3, E_3 and F_3, respectively.



Since the graph has fewer than n edges, we have that $|V_3| + |F_3| = |E_3| + 1$. The graph H still has the same vertex set as G, hence $|V| = |V_3|$.

We have removed only one edge, so it is clear that $|E_3| = |E| - 1$. The removal of this edge has caused one face/region to disappear, thus $|F_3| = |F| - 1$.

Combining these we find that

$$|V| + |F| = |V_3| + |F_3| + 1 = |E_3| + 1 + 1 = (|E_3| + 1) + 1 = |E| + 1.$$

This has shown us that it must be true for all connected planar graphs with n edges. By the principle of (strong) induction, the result is true for all connected planar graphs. $\square$

- Bondy & Murty, *Graph Theory with Applications*. Proves Euler's Formula Using induction on faces.
- Twenty proofs of Euler's formula: `http://www.ics.uci.edu/~eppstein/junkyard/euler/`

Some related problems:

Question (Turkish Math. Olympiad 1993)

Some towns are connected to each other by some roads with at most one road between any pair of towns. Let v denote the number of towns, and e the number of roads. Show that

a) *if $e < v - 1$, then there are at least two towns such that it is impossible to travel from one to the other,*

b) *if $2e > (v-1)(v-2)$, then travelling between any pair of towns is possible.*

Question (7th Irish Mathematical Olympiad)

If a square is partitioned into n convex polygons, determine the maximum number of edges present in the resulting figure. [Hint: By Euler's theorem, if the square is partitioned into n polygons, then $v - e + n = 1$ where v is the number of vertices and e is the number of edges in the resulting figure]

2. The Pigeonhole Principle

Pigeonhole Principle I: If $n > m$ pigeons are put into m holes, then some hole will contain more than 1 pigeon.

Example

If 11 letters are to be distributed into 10 letter-boxes, then one letter box contains at least two letters.

Pigeonhole Principle II: If $n > m$ pigeons are put into m holes, then some hole will contain at least $\lceil n/m \rceil$ pigeons.

Example

In Dublin, there are at least seven people with exactly the same number of hairs on their heads. The population of Dublin is 1,800,001. An upper bound for the number of hairs on a human head is 300,000 (the average being 100,000). In this situation, we have 300,000 boxes (i.e. pigeonholes) into which we place the names of the 1,800,001 people according to how many hairs they have. Therefore, there must be one box containing at least $\lceil 1800001/300000 \rceil = 7$ names.

Problem

Suppose we have two columns and ten rows. We place the numbers from the set $\{1, 2, \ldots, 20\}$ *into the twenty boxes such that*

- *there is one odd and one even number in every row,*
- *the first column contains number from the set*

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\},$$

- *the second column contains numbers from the set*

$$\{11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}.$$

Show that there are two rows with the exact same sum.

Example

5	12
2	13
3	14
4	15
1	18
6	17
7	16
8	19
9	20
10	11

Form the array of all possibilities:

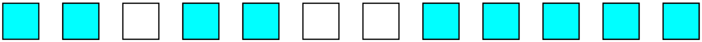
	1	2	3	4	5	6	7	8	9	10
11		13		15		17		19		21
12	13		15		17		19		21	
13		15		17		19		21		23
14	15		17		19		21		23	
15		17		19		21		23		25
16	17		19		21		23		25	
17		19		21		23		25		27
18	19		21		23		25		27	
19		21		23		25		27		29
20	21		23		25		27		29	

Notice that regardless of the way the numbers are placed, the sum in each row can be one of only nine numbers: $13, 15, \ldots, 29$.

Since there are ten rows, there will be ten sums, and by the pigeonhole principle at least two sums must be the same. $\square$

Nine people are seated in a row of 12 chairs. Show there are three consecutive chairs which are filled.

Consider a particular instance of this:



Since there are 9 people seated in 12 chairs, 3 chairs are always empty. Let us condition on the empty chairs:

The general situation is

$$\mathbf{A} \ \boxed{\text{e1}} \ \mathbf{B} \ \boxed{\text{e2}} \ \mathbf{C} \ \boxed{\text{e3}} \ \mathbf{D}$$

where A, B, C, and D represent rows of seats with people sitting beside one-another.
All 9 people must be in one of A, B, C, and D, so by the pigeonhole principle one of these 'boxes' contains at least $\lceil 9/4 \rceil = 3$ people.

That is, in any such configuration there are **at least** three consecutive chairs which are filled. $\square$.

Six integers are chosen from the set $A := \{1, 2, 3, \ldots, 10\}$. Show that some two of them will have an odd sum.

Working 'backwards', notice the sum of two integers if odd $\Leftrightarrow$ one is odd and one is even:

a	b	$a+b$
odd	odd	even
odd	even	odd
even	odd	odd
even	even	even

So it will be sufficient to show that among any six numbers chosen from A, at least one odd and one even integer appears:

This is clear since there are precisely 5 odd and 5 even numbers in the set A. When choosing six there must be at least one of each. (This follows by the P-H principle since there are 4 numbers not chosen, and conditioning on these there are 5 groups into which to place 6 numbers, $\Rightarrow$ one box contains at least 2 numbers $\Rightarrow$ two consecutive numbers $\Rightarrow$ 1 odd & 1 even.) $\square$.

55 distinct integers are selected from the set $\{1, 2, 3, 4, \ldots, 100\}$. Show that there must be some pair of these which differ by 9.

Consider the partition of the integers from this set into classes which contain numbers which differ by 9.

Class	Numbers	#
(a)	1 – 10 – 19 – 28 – 37 – 46 – 55 – 64 – 73 – 82 – 91 – 100	12
(b)	2 – 11 – 20 – 29 – 38 – 47 – 56 – 65 – 74 – 83 – 92	11
(c)	3 – 12 – 21 – 30 – 39 – 48 – 57 – 66 – 75 – 84 – 93	11
(d)	4 – 13 – 22 – 31 – 40 – 49 – 58 – 67 – 76 – 85 – 94	11
(e)	5 – 14 – 23 – 32 – 41 – 50 – 59 – 68 – 77 – 86 – 95	11
(f)	6 – 15 – 24 – 33 – 42 – 51 – 60 – 69 – 78 – 87 – 96	11
(g)	7 – 16 – 25 – 34 – 43 – 52 – 61 – 70 – 79 – 88 – 97	11
(h)	8 – 17 – 26 – 35 – 44 – 53 – 62 – 71 – 80 – 89 – 98	11
(i)	9 – 18 – 27 – 36 – 45 – 54 – 63 – 72 – 81 – 90 – 99	11

If we choose 55 numbers from $\{1, \ldots, 100\}$, by the P-H principle, one of the 9 classes must contain at least $\lceil 55/9 \rceil = 7$ of these numbers.

There are 2 cases to consider:

(I) if the class is (a), and
(II) if it is one of the classes (b)–(i).

For (I), if at least 7 of these numbers are in (a), then by using the same application of the P-H principle as we did in the chairs problem, it is easy to see there must be two adjacent entries $\Rightarrow$ two of the 55 numbers differ by exactly nine.

For (II), the same technique is used, and we see that if 7 numbers are chosen from 11 (e.g. case(f)), then there must be two adjacent entries. $\square$.

Theorem

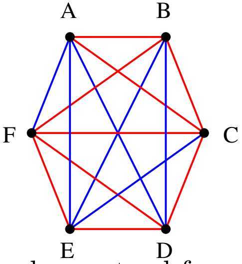
In a group of six people, there are three people who mutually know each other, or mutually do not know each other.

Proof: We first assume that knowing and not knowing are commutative concepts.

Let the six people be called A, B, C, D, E and F.

Associate a graph with vertices's A, B, C, D, E, F with this situation in which two people are connected by a blue line if they mutually know one another and a red one if they do not.

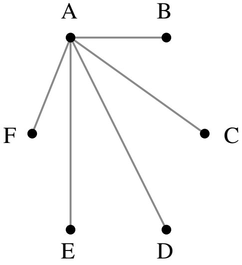
For example here is one such possibility:



In light of this graph-theoretic set-up, we want to prove that for any colouring of the edges (red or blue) of this graph, there will always be at least one red or blue triangle.

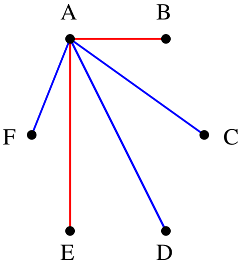
Consider person (vertex) A.

5 edges extend from this vertex and each of the edges must be coloured either red or blue.

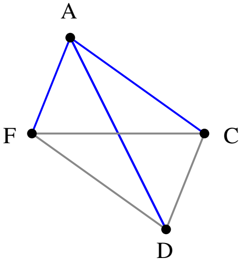


By the pigeonhole principle, since we have 5 lines and 2 colours from which to colour each of these lines, there must be at least $\lceil 5/2 \rceil = \lceil 2.5 \rceil = 3$ lines of the same colour.

Without loss of generality, let us assume these lines are blue and the corresponding end points are C, D and F.

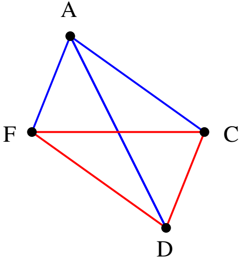


Removing the points B and E we see that we still must colour each of the three edges CF, CD and DF either red or blue.



If we colour any of these edges blue then we will have completed a blue triangle.

Since we do not wish to do this, none of them can be blue and so must be red.



However, this gives us a red triangle between the points C, D, and F. So it is impossible to colour the edges in such a way that no red or blue triangle is formed.

Thus, in any group of six people, there exist three that either mutually know, or don't know, one another. $\square$

Problem (Sixth IMO, 1964, Problem 3)

Seventeen people correspond by mail with one another – each one with all the rest. In their letters only three topics are discussed. Each pair of correspondents deals with only one of these topics. Prove that there are at least three people who write to each other about the same topic.

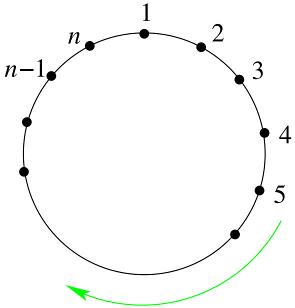
Problem

Each square of a 3×7 *board is coloured black or white. Prove that for any such colouring, the board contains a rectangle whose 4 corners are all of the same colour.*

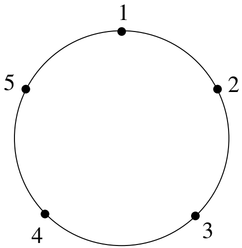
Setup:

- 41 people standing in a circle waiting to be executed, every third person gets executed by the third person ahead (and still alive!) in a clockwise direction, the last killer is then killed by the same rule etc.
- Who is the last man standing?

We consider the case of n men (vertices) around a circle with vertices eliminated in the order $2, 4, \ldots$



So for the $n = 5$ case we find ....



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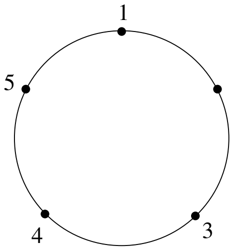

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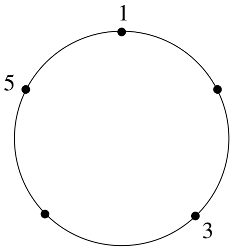

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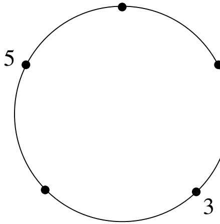

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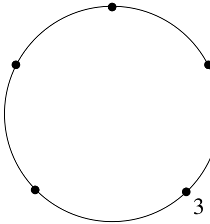
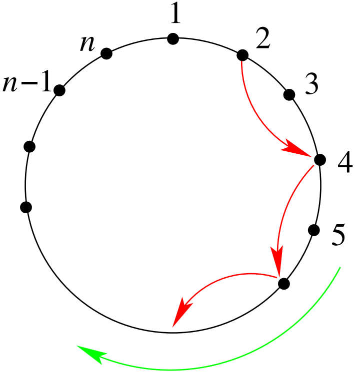


So for the $n = 5$ case we find .... that vertex number 3 remains.

What is the answer for general n ?

For the general problem problem on the circle with n vertices, where every second vertex is removed in a clockwise direction beginning at 2, let us denote the number remaining by $J(n)$.



$\boxed{\text{We want a formula/expression for } J(n)}$

How do we find such a formula? Maybe try ...

- Calculating the first few values and see if there's a pattern.
- Finding some kind of recursion.
- Removing a vertex and see how that changes things.
- Look at what happens when n is odd and n is even.

The first few values: Examining the cases for n small, one finds:

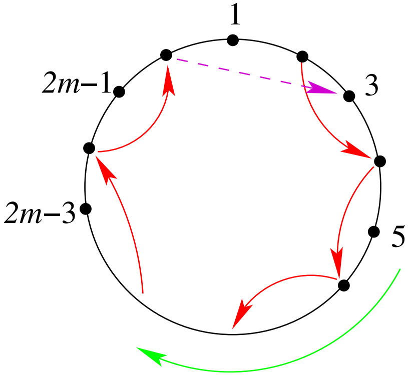
n	1	2	3	4	5	6
$J(n)$	1	1	3	1	3	5

There could be some sort of pattern but the answer doesn't seem obvious.

Looking for a recursion:

Consider the cases when n is odd and even separately.

- If $n = 2m$, then after going once around the circle, we find that all the even numbers are gone.

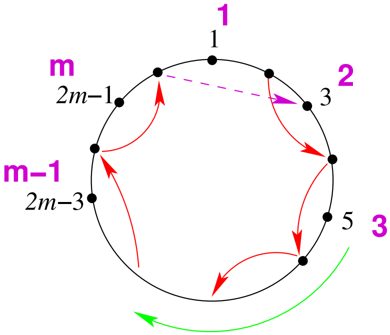


Who is next to go?

Answer: Vertex 3.

Notice that if we relabel to the purple numbers, it is the same as beginning the problem again at 2 and having m vertices.

How do we go from the purple labels to the original labels??



$$\text{original} \;=\; 2 \times (\text{purple}) - 1$$

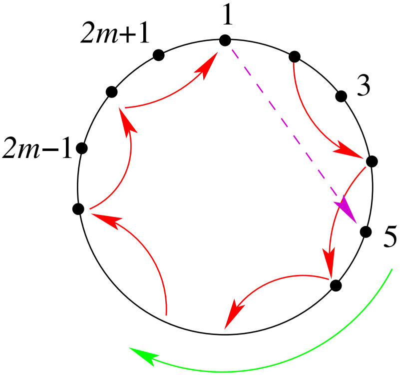
So if $J(m)$ is the last vertex remaining of the purple numbers, then the corresponding original number is simply $2J(m) - 1$.

$$\boxed{J(2m) = 2J(m) - 1}$$

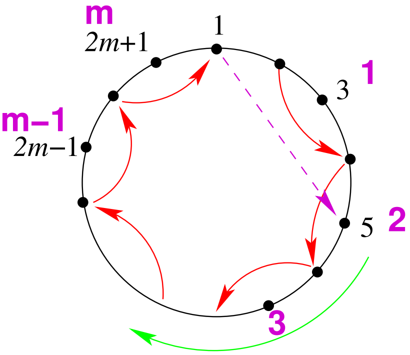
- What about $n = 2m + 1$?

  This is almost the same as the n is even case.

  Go once around the circle to find $2, 4, \ldots, 2n, 1$ have been removed and the next vertex which is due to be removed is 5.



Inserting new labels we see this to be now equivalent to the problem on m vertices.



How do we go from the purple labels to the original labels in this case??

$$\text{original} \;=\; 2\times(\text{purple})+1$$

This gives

$$\boxed{J(2m+1)=2J(m)+1}$$

- We now have a method for computing the numbers $J(n)$ without arguing our way around (and in) circles:

$$
\begin{aligned}
J(1) &= 1, \\
J(2m) &= 2J(m) - 1, \text{ for } m \geq 1, \\
J(2m+1) &= 2J(m) + 1, \text{ for } m \geq 1.
\end{aligned}
$$

Can we find a nicer formula?
Compute the values:-

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$J(n)$	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1	3	5	7	9

$$\downarrow$$

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$J(n)$	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1	3	5	7	9

From this, we may conjecture:-

If $n = 2^m + k$ where $0 \le k < 2^m$,
then $J(n)$ is the $(k+1)^{th}$ odd number,
i.e.

$$J(n) \;=\; 2k+1.$$

Q1: Can you find a nice interpretation on $J(n)$ in terms of the binary representation of n?

Q2: What about when every third person is eliminated, beginning with person numbered 3? Does the same analysis provide a solution?

Question

Prove that $\dfrac{(2m)!(2n)!}{m!n!(m+n)!}$ *is an integer for all integers* $m, n \ge 0$.

4. Permutations and their subsequences

Theorem (Erdös-Szekeres theorem)

Let $a_1, a_2, \ldots, a_{n^2+1}$ *be a permutation of the set* $\{1, 2, \ldots, n^2+1\}$. *There exists a subsequence of this sequence*

$$a_{i(1)}, a_{i(2)}, \ldots, a_{i(n)}, a_{i(n+1)}$$

which is either increasing $a_{i(1)} < a_{i(2)} < \cdots < a_{i(n)} < a_{i(n+1)}$, *or decreasing* $a_{i(1)} > a_{i(2)} > \cdots > a_{i(n)} > a_{i(n+1)}$.

Example

Consider the sequence of $10 = 3^2 + 1$ numbers;

$$7, 3, 8, 1, 2, 6, 4, 5, 10, 9.$$

In this case $a_1 = 7$, $a_2 = 3$, ..., $a_{10} = 9$. The above claim says there is a subsequence of length 4 $(= 3 + 1)$ in this sequence which is either increasing or decreasing. Evidenced by $1, 2, 4, 9$ which is a_4, a_5, a_7, a_{10}.

Proof.

For each of the numbers a_j with $1 \leq j \leq n^2 + 1$, let us associate a number t_j which tells us the length of the longest increasing subsequence starting at position j. In the example just given,

j	1	2	3	4	5	6	7	8	9	10
a_j	7	3	8	1	2	6	4	5	10	9
t_j	3	4	2	5	4	2	3	2	1	1

If there is some value j, $1 \leq j \leq n^2 + 1$, such that $t_j \geq n + 1$ then we are done since this means there is an increasing subsequence beginning with the value a_j in our sequence.

[In the example, since $t_5 = 4 (= n + 1)$ we can see the existence of an increasing subsequence of length 4]

$\square$

Proof cont'd

Assume this is not the case; so $t_j \leq n$ for all $1 \leq j \leq n^2 + 1$.
For each of the $n^2 + 1$ numbers in the sequence, we have $t_j \leq n$.
Consider n boxes labelled $B_1, \ldots, B_n$ into which we place the elements of our sequence $a_1, a_2, \ldots a_{n^2+1}$ according to the following rule

$$\boxed{\text{Place } a_k \text{ in box } B_j \text{ if } t_k = j.}$$

i.e., put an element from the sequence into the box whose label gives the longest increasing subsequence beginning with that element.
We have a total of $n^2 + 1$ numbers to place into n boxes.

By the P-H principle, at least one of these boxes must contain

$$\left\lceil \frac{n^2+1}{n} \right\rceil = n + 1 \text{ numbers.}$$

Let this box have label s.

Proof cont'd

Then there exists a subsequence $a_{i(1)}, a_{i(2)}, \ldots, a_{i(n)}, a_{i(n+1)}$ such that

$$t_{i(1)} = t_{i(2)} = \ldots = t_{i(n)} = t_{i(n+1)} = s.$$

Final step: Consider the subsequence in question now:

$$a_{i(1)}, a_{i(2)}, \ldots, a_{i(n)}, a_{i(n+1)}.$$

If $a_{i(1)} < a_{i(2)}$ then, since there is an increasing sequence of length s beginning with $a_{i(2)}$, and since $a_{i(1)} < a_{i(2)}$, an increasing sequence of length (at least) $s + 1$ begins at $a_{i(1)}$.

In our notation, this means that $t_{i(1)} \geq s + 1$.

But this is a contradiction since $t_{i(1)} = s$ so we must have $a_{i(1)} > a_{i(2)}$.

This exact same argument shows that $a_{i(2)} > a_{i(3)}$, $a_{i(3)} > a_{i(4)}$, etc.

Proof cont'd

Putting these inequalities together yields

$$a_{i(1)} > a_{i(2)} > \cdots > a_{i(n+1)}$$

which shows the existence of a length $n+1$ decreasing subsequence. $\square$

Note: This result does not hold for subsequences of length $n+2$.

Exercise: Find a permutation of $\{1, 2, \ldots, 10\}$ which contains no increasing or decreasing subsequence of length 5.

Question

Prove that the set $S = \{1, 2, \ldots, 1989\}$ can be expressed as the disjoint union of subsets A_i (for $i = 1, 2, \ldots, 117$) such that

(i) *each A_i contains 17 elements*

(ii) *the sum of the elements in each A_i is the same.*

($A = A_1 \cup A_2 \cup \cdots \cup A_{117}$ with $A_i \cap A_j = \emptyset$ for all i, j with $i \neq j$)