Mathematical Enrichment Programme
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Complex numbers

A *complex number* is a "number" of the form $a + bi$ where a, b are real numbers, and i is a symbol satisfying the relation $i^2 = -1$. If $z = a + bi$ then the numbers a and b are called the *real part* and the *imaginary part* of the complex number z respectively (notation $a = \operatorname{Re} z$, $b = \operatorname{Im} z$); number $a - bi$ is called *the conjugate* of z (notation $\overline{z}$). Two complex numbers can be added

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

and multiplied (using the formula $i^2 = -1$)

$$(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i\,.$$

If $c + di \neq 0$ (meaning that at least one of numbers c, d is non-zero), we can divide any complex number by $c + di$:

$$\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}\,i\,.$$

We see that indeed complex numbers deserve to be called numbers: just as for reals, we can add, subtract, multiply and divide by non-zero numbers.

Exercises • Check that $\operatorname{Re} z = (z + \overline{z})/2$ and $\operatorname{Im} z = (z - \overline{z})/2i$. • Check that $z = 1$, $z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$ are solutions to the equation $z^3 = 1$ in complex numbers.

We can "draw" a complex number $z = a + bi$ as point (a, b) on the plane with fixed system of coordinates. The conjugate $\overline{z} = a - bi$ is then the mirror reflection of z with respect to the horizontal axis. The non-negative real number $|z| = \sqrt{a^2 + b^2}$, the distance from z to 0, is called *the absolute value* of z.

Polar form and Euler's formula

For every non-zero complex number z we have $\operatorname{Re} z = r \cos\phi$ and $\operatorname{Im} z = r \sin\phi$ where $r = |z|$ and ϕ is the angle between the positive direction of the real axis and the ray from 0 to z. The representation $z = r(\cos\phi + i \sin\phi)$ is called the *polar form*. The angle ϕ is called the *argument* of z. When we multiply two complex numbers

$$\begin{aligned} z_1 z_2 &= r_1(\cos\phi_1 + i\sin\phi_1) r_2(\cos\phi_2 + i\sin\phi_2) \\ &= r_1 r_2((\cos\phi_1\cos\phi_2 - \sin\phi_1\sin\phi_2) + i(\cos\phi_1\sin\phi_2 + \sin\phi_1\cos\phi_2)) \\ &= r_1 r_2(\cos(\phi_1 + \phi_2) + i\sin(\phi_1 + \phi_2))\,, \end{aligned}$$

we see that their absolute values multiply but their arguments add.

Exercise • The exponential function of a complex number is defined as $e^z = e^{a+bi} = e^a(\cos b + i \sin b)$. Show that this function enjoys the property $e^{z_1+z_2} = e^{z_1}e^{z_2}$ for any two complex numbers z_1, z_2. (Assume that this property for the exponential with real arguments is known.) • Deduce from the previous exercise *de Moivre's formula* $\cos(n\phi) + i\sin(n\phi) = (\cos\phi + i\sin\phi)^n$ and use it to get the usual trigonometric formulas for $\cos(2\phi)$, $\sin(2\phi)$, $\cos(3\phi)$, etc.

The formula $e^{i\phi} = \cos\phi + i\sin\phi$ for real ϕ is known as *Euler's formula*. The polar form of a complex number can be written as $z = re^{i\phi}$.

Complex solutions to the equation $z^n = 1$ are called nth roots of unity. They are given by $z = e^{\frac{2\pi k}{n}i}$ with $k = 0, 1, 2, \ldots, n-1$. As usual, we can factorize a polynomial when we know its roots:

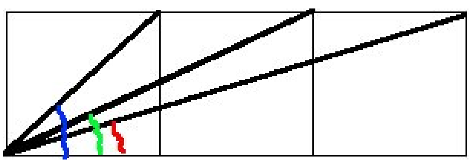
$$x^n - 1 = \prod_{k=0}^{n-1}\left(x - e^{\frac{2\pi k}{n}i}\right).$$

Exercise • Check this formula explicitly when $n = 3$. Notice that $e^0 = e^{2\pi i} = 1$, $e^{\frac{2\pi i}{3}} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $e^{\frac{4\pi i}{3}} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$. Draw the 3rd roots of unity.

Since $\prod_{k=0}^{n-1}\left(x - e^{\frac{2\pi k}{n}i}\right) = x^n - (\sum_{k=0}^{n-1} e^{\frac{2\pi k}{n}i})x^{n-1} + \ldots + (-1)^n \prod_{k=0}^{n-1} e^{\frac{2\pi k}{n}i}$ is equal to $x^n - 1$, we see that the sum of the nth roots of unity is equal to 0 and their product is equal to $(-1)^{n+1}$. The nth roots of unity lie on the unit circle and joining them one gets a regular n-gon.

Complex numbers help geometry

- The three squares on the image below are equal. Compute the sum of the three angles (blue, green and red).



- For a rectangle $ABCD$ and any point P in the plane, prove that $|PA|^2 + |PC|^2 = |PB|^2 + |PD|^2$.

- Describe how to construct a regular pentagon using ruler and compass. *Hint: Whenever you see a regular n-gon, it is convenient to assume that its vertices are located at nth roots of unity.*

- Prove that the centroid of a triangle with vertices at complex numbers a, b and c is given by $m = (a+b+c)/3$. *Hint: first observe that for two complex numbers z, w and any positive real α, β with $\alpha + \beta = 1$ the number $\alpha z + \beta w$ lies on the segment joining z and w and divides it at the ratio $\beta : \alpha$. For example, the centre of the segment is given by $(z+w)/2$.*

- (Napoleon's Theorem) We build equilateral triangles on the sides of a given triangle (on the outside, so that the three new triangles share only edges with the given triangle). Prove that the centroids of the three new triangles form an equilateral triangle. What could you say about the centroid of this triangle?

- Consider a triangle with vertices at complex numbers a, b and c. Prove that its orthocenter h and circumcenter o satisfy $h + 2o = a + b + c$. *Hint: use Euler's line.*

- Let $ABCD$ be a cyclic quadrilateral and H_A, H_B, H_C and H_D be the orthocentres of triangles BCD, CDA, DAB and ABC respectively. Prove that quadrilaterals $ABCD$ and $H_AH_BH_CH_D$ are congruent. *Hint: Whenever in a given problem there is a circle, it is often convenient to assume that the centre of this circle is located at* 0.

- Let a and b lie on the unit circle and let p be the point of intersection of the tangent lines to the unit circle at a and b. Prove that

$$p = \frac{2ab}{a+b}.$$

Hint: Observe that the transformation $z \mapsto \frac{1}{\bar{z}}$ *is the inversion with respect to the unit circle.*

- A quadrilateral $ABCD$ is circumscribed on a circle and the sides AB and CD are parallel. The side DA is tangent to the inscribed circle at E. Point F is the reflection of A with respect to B. The line tangent to the inscribed circle, passing through F and distinct from AB touches the circle at G. Prove that C, E and G are collinear.

- Let $A_0A_1A_2A_3A_4A_5A_6$ be regular 7-gon. Prove that

$$\frac{1}{|A_0A_1|} = \frac{1}{|A_0A_2|} + \frac{1}{|A_0A_3|}.$$

- On the sides of a parallelogram we build four squares. Prove that the centres of these squares form a square.

- Prove that if a quadrilateral is circumscribed on a circle, then the diagonals and the segments connecting opposite tangency points all pass through one point.

- Points P, A, B, C are distinct and lie on a circle. Prove that the projections of P on the lines AB, BC, CA are collinear (this line is called the Simson line).

More on roots of unity

- Calculate the sum

$$\sum_{k\geq 0} \binom{n}{3k} = \binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \ldots + \binom{n}{\lfloor n/3 \rfloor}.$$

Hint: Try to substitute 3*rd roots of unity into the sum* $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$.

- Let $z_k = e^{\frac{2\pi k}{n} i}$ with $k = 0, 1, 2, \ldots, n-1$ be all nth root of unity. Show that

$$z_0^m + z_1^m + \ldots z_{n-1}^m = \begin{cases} n & \text{when } n \text{ divides m} \\ 0 & \text{otherwise} \end{cases}$$

- We say that a complex number z is a *primitive* nth root of unity if $z^n = 1$ but $z^m \neq 1$ for any $0 < m < n$. For $n > 1$, show that primitive nth roots of unity are given by $z_k = e^{\frac{2\pi k}{n} i}$ where $1 \leq k < n$ is comprime to n.