0)

Centres of Triangles

28th February 2015

Here are the notes I said I would put online for you. They include:

1) The final three points of the nine-point circle which we did quickly on the blackboard,

2) A proof that the centre N of this circle lies on the Euler line,

3) Some diagrams that might be of interest,

4) A list of popular maths books you might enjoy reading.

I hope you all continue to enjoy maths and that you come back next year for more maths enrichment if you're still in school.

Best wishes,

Mary Hanley

Note: We will continue the same notation:

$[AA']$, $[BB']$ and $[CC']$ denote the medians of $\triangle ABC$,

$[AA^*]$, $[BB^*]$, $[CC^*]$ " " altitudes " ",

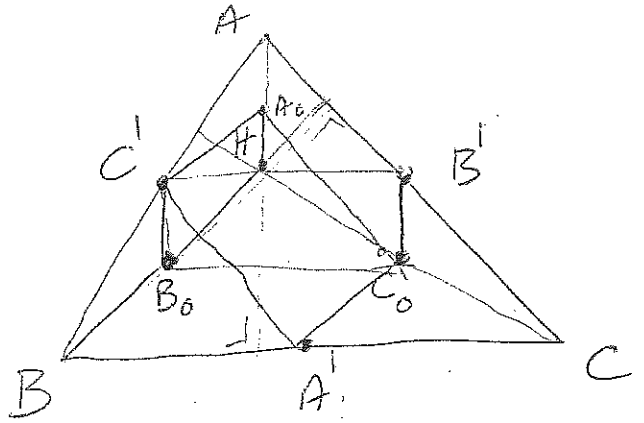
O is the circumcircle of " ",

G is the centroid of " ",

H is the orthocentre of " ".

1)

The Nine-point circle.



A_0 is the midpoint of $[AH]$
B_0 " - - - of $[BH]$
C_0 - - - - - $[CH]$.

We will show that A_0, B_0, C_0 are on the same ⊙ as A', B', C' (with $A^*, B^*,$ and C^* making nine points on the same circle).

Consider $\triangle BAH$ and $\triangle CAH$.
$C'B_0 \parallel AH \parallel B'C_0$ since $|AC'| = |C'B|$ and $|HB_0| = |B_0B|$
and $|AB'| = |B'C|$ and $|HC_0| = |C_0C|$.

Also $AH \perp BC$ and $B_0C_0 \parallel BC$
So $C'B'C_0B_0$ is a rectangle and so is cyclic with diameter $[C'C_0]$ in its circumcircle.

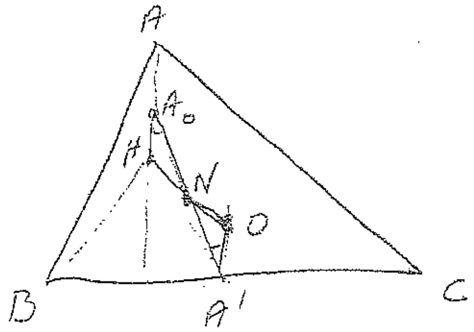
Similarly, $C'A_0 \parallel A_1C_0$ and $C'A_1 \parallel A_0C_0$. Thus, $C'A_0C_0A_1$ is a rectangle and so cyclic with the same diameter $[C'C_0]$.

Thus all six points $A_0, B_0, C_0, A', B', C'$ lie on the same circle. We already proved that A^*, B^* and C^* also lie on this circle. We call it the nine-point circle (OR the circumcircle of the medial $\triangle$).
(It is sometimes referred to as Feuerbach's circle.)

2)

The Euler Line again

We will prove that the "triangle centre" N, that is the centre of the nine point circle, also lies on the Euler line.



From the previous page, we know that $[C_0C']$, $[A_0A']$ and $[B_0B']$ are diameters of the nine-point circle. So N is the midpoint of, say, $[A_0A']$.

Consider the $\triangle A_0HN$ and the $\triangle A'ON$.

Now, $|A_0H| = |OA'|$, (see * below)

$|A_0N| = |A'N|$ and $|\angle HA_0N| = |\angle NA'O|$ $(AH \parallel OA')$

so these $\triangle$s A_0HN and $A'ON$ are congruent.

Thus $|\angle HNA_0| = |\angle ONA'|$. We know that A_0, N and A' are collinear, so these are truly opposite angles. Therefore H, N and O must be collinear. That is, N lies on the Euler line.

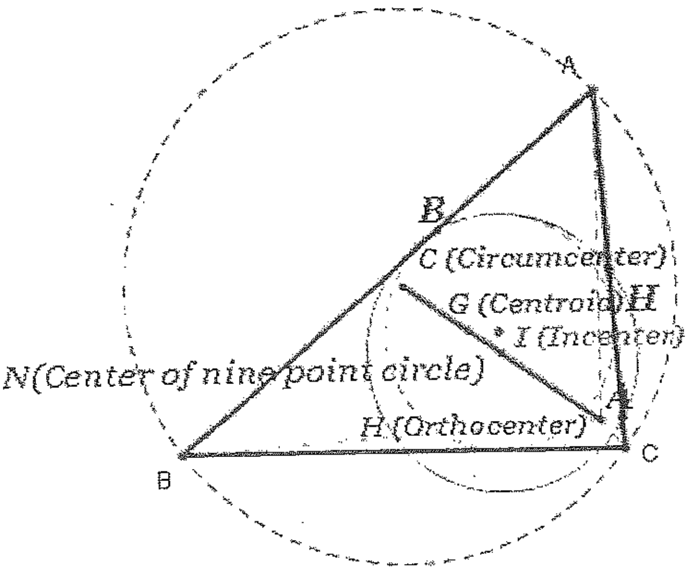
Recall that $3|OG| = |OH|$.

We also have from the above that $|HN| = |NO|$, so, $2|ON| = |OH| = 3|OG|$. The line segments connecting the points of the Euler line that we've met, are separated by lengths in small natural number ratios to each other.

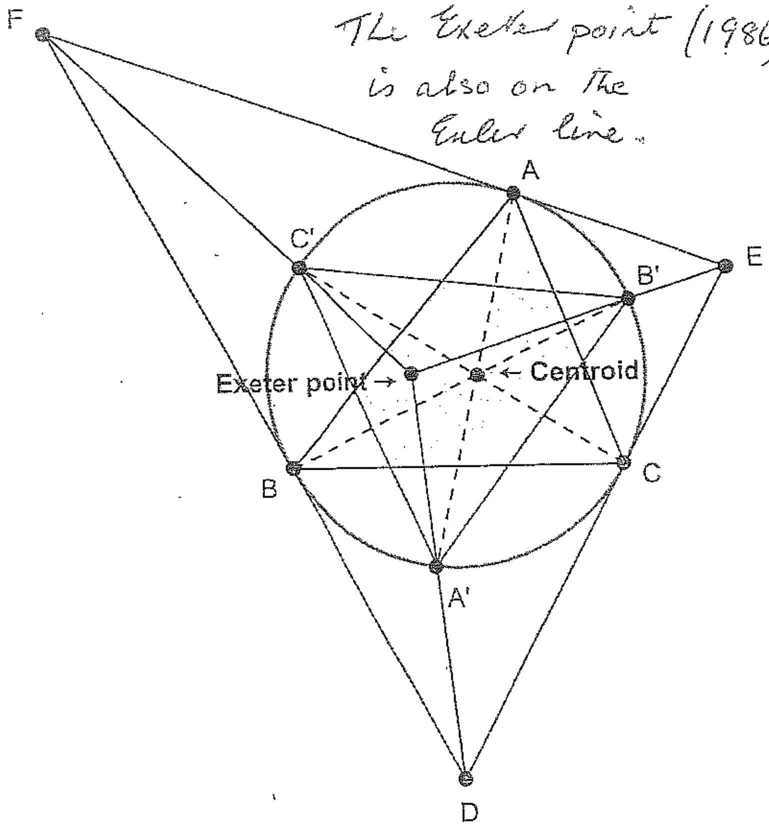
* Recall that when we showed that the points H, G and O are collinear (on the Euler line), we proved that the $\triangle$s AHG and OGA' are similar. We noted that $|AH| = 2|OA'|$ which was irrelevant then, but we use it here.

3)

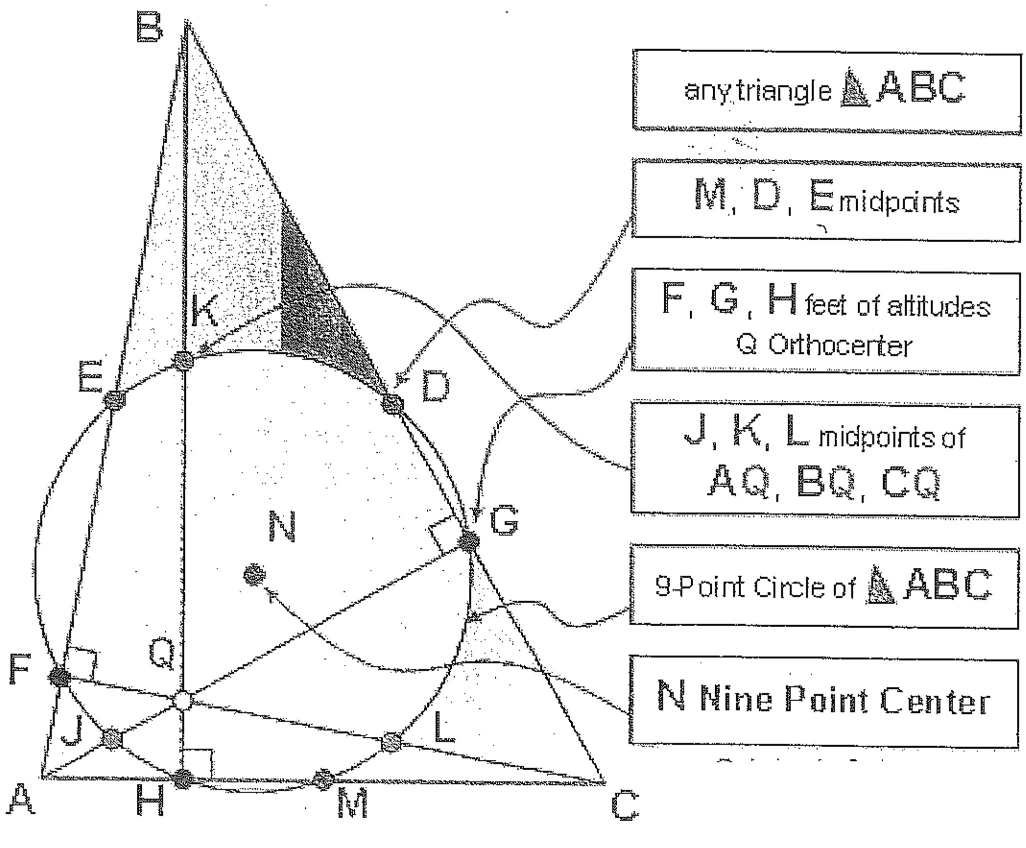
The Euler Line



The Exeter point (1986) is also on the Euler line.



The Ninepoint Circle



4)

Popular maths books

Mary Hanley

The books that I've listed here can be enjoyed by people who have an interest in maths without having studied maths at third level.

- *Alex's Adventures in Numberland* (2010), by Alex Bellos,
- *Fermat's Last Theorem* (1997), by Simon Singh,
- *The Code Book: The Science of Secrecy from Ancient Egypt to Quantum Cryptography* (2000), by Simon Singh,
- *The Music of the Primes* (2003), by Marcus du Sautoy,
- *Prime Obsession: Bernhard Riemann and the Greatest Unsolved Problem in Mathematics* (2003), by John Derbyshire.