Inequalities

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The Arithmetic Mean – Geometric Mean (AM-GM) Ineq.

For any two positive real numbers a and b, we have

$$\frac{a+b}{2} \geq \sqrt{ab}$$

with equality if and only if $a = b$.

More generally, for n positive real numbers $x_1, x_2, \ldots, x_n$ we have

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}$$

with equality if and only if all of the numbers $x_1, x_2, \ldots, x_n$ are equal.

Example 1. Show that for all positive integers $n \geq 2$ we have

$$n! < \left(\frac{n+1}{2}\right)^n.$$

Solution. The AM-GM inequality gives

$$1 + 2 + \cdots + n > n\sqrt[n]{1 \cdot 2 \cdots n} = n\sqrt[n]{n!}.$$

Therefore

$$\frac{n(n+1)}{2} = 1 + 2 + \cdots + n > n\sqrt[n]{n!}.$$

Cancelling n on both sides gives

$$\frac{n+1}{2} > \sqrt[n]{n!}$$

and taking n-th powers gives the required inequality.

Example 2. Let $a, b, c > 0$ be such that

$$(1 + a)(1 + b)(1 + c) = 8.$$

Prove that $abc \leq 1$.

Solution. Assume that $abc > 1$. By AM-GM inequality we have

$$1 + a \geq 2\sqrt{a}$$

$$1 + b \geq 2\sqrt{b}$$

$$1 + c \geq 2\sqrt{c}$$

We now multiply side by side the above inequalities.

Using $abc > 1$ we find

$$(1 + a)(1 + b)(1 + c) \geq 8\sqrt{abc} > 8,$$

contradiction. Hence, $abc \leq 1$.

Example 3. (a) Prove that for any positive numbers x, y we have

$$x^3 + y^3 \geq x^2y + xy^2.$$

(b) Prove that for any real numbers $0 \leq x, y, z \leq 1$ we have

$$3 + x^3 + y^3 + z^3 \geq x^2 + y^2 + z^2 + x + y + z.$$

Solution. (a) We have

$$x^3 + y^3 - (x^2y + xy^2) = (x + y)(x - y)^2 \geq 0.$$

(b) Using the above inequality we have

$$2+x^3+y^3 \geq 2+x^2y+xy^2 = (1+x^2y)+(1+xy^2) \geq (x^2+y)+(y^2+x).$$

(*) students required details in order to prove that

$$(1 + x^2y) + (1 + xy^2) \geq (x^2 + y) + (y^2 + x)$$

Similarly we have

$$2 + y^3 + z^3 \geq z^2 + y + y^2 + z$$

$$2 + z^3 + x^3 \geq z^2 + x + x^2 + z.$$

Adding the above three inequalities we obtain the conclusion.

Example 4. (a) Prove that for any positive real numbers a, b we have

$$\frac{a+b}{2} \geq \frac{2}{\frac{1}{a}+\frac{1}{b}}.$$

(b) Prove that for positive real numbers x, y, z,

$$\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x} \leq \frac{1}{2}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right).$$

Solution. (a) The inequality is equivalent to

$$\frac{a+b}{2} \geq \frac{2ab}{a+b}$$

or even $(a+b)^2 \geq 4ab$ which can be written $(a-b)^2 \geq 0$. We have equality if and only if $a = b$.

(b) We apply the above inequality for $a = \frac{1}{x}$ and $b = \frac{1}{y}$. We have

$$\frac{1}{2}\left(\frac{1}{x}+\frac{1}{y}\right) \geq \frac{2}{x+y}. \qquad (1)$$

Similarly we obtain

$$\frac{1}{2}\left(\frac{1}{y}+\frac{1}{z}\right) \geq \frac{2}{y+z}, \qquad (2)$$

$$\frac{1}{2}\left(\frac{1}{z}+\frac{1}{x}\right) \geq \frac{2}{z+x}. \qquad (3)$$

Adding up the inequalities (1) (2) and (3) we find

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \geq \frac{2}{x+y}+\frac{2}{y+z}+\frac{2}{z+x}$$

which proves our inequality.

Example 5. Prove that for any positive real numbers a, b and c we have

$$\frac{2a+b}{b+2c}+\frac{2b+c}{c+2a}+\frac{2c+a}{a+2b}\geq 3.$$

Solution. Let

$$b+2c=x \qquad (1)$$

$$c+2a=y \qquad (2)$$

$$a+2b=z \qquad (3)$$

Adding the above equalities we find

$$2a+2b+2c=\frac{2(x+y+z)}{3} \qquad (4)$$

Now, from (1) and (4) we find

$$2a+b=\frac{2y+2z-x}{3}$$

and similarly,

$$2b+c=\frac{2x+2z-y}{3} \qquad \text{and} \qquad 2c+a=\frac{2x+2y-z}{3}.$$

Thus, in the new variables x, y, z our initial inequality reads

$$\frac{1}{3}\left\{\frac{2y+2z-x}{x}+\frac{2x+2z-y}{y}+\frac{2x+2y-z}{z}\right\}\geq 3,$$

or even

$$2\left(\frac{x}{y}+\frac{y}{x}\right)+2\left(\frac{y}{z}+\frac{z}{y}\right)+2\left(\frac{x}{z}+\frac{z}{x}\right)\geq 12. \qquad (5)$$

By AM-GM inequality we have

$$\frac{x}{y} + \frac{y}{x} \geq 2, \quad \frac{y}{z} + \frac{z}{y} \geq 2, \quad \frac{x}{z} + \frac{z}{x} \geq 2.$$

Adding the above inequalities we find (5) which proves our initial inequality.

Example 6. The non-zero real numbers a, b, c, d satisfy the equalities

$$a + b + c + d = 0, \quad \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{abcd} = 0.$$

Find, with proof, all possible values of the product $(ab - cd)(c + d)$.

Solution. From

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{abcd} = 0$$

we deduce that

$$bcd + cda + dab + abc = -1 \ .$$

So,

$$-1 = bcd + cda + dab + abc = cd(b + a) + ab(c + d) \ .$$

Using the fact that $a + b = -(c + d)$ yields

$$-1 = (c + d)(ab - cd)$$

and so $(ab - cd)(c + d) = -1$ for all admissible values of a, b, c, d.

Example 7. Let $a, b, c > 0$ be such that $abc = 1$. Prove that

$$\frac{1+ab}{1+a} + \frac{1+bc}{1+b} + \frac{1+ca}{1+c} \geq 3.$$

Solution. Observe first that

$$\frac{1+ab}{1+a} = \frac{abc+ab}{1+a} = ab\frac{1+c}{1+a}.$$

Similarly,

$$\frac{1+bc}{1+b} = bc\frac{1+a}{1+b}, \quad \frac{1+ca}{1+c} = ca\frac{1+b}{1+c}.$$

By AM-GM inequality we now obtain

$$\frac{1+ab}{1+a} + \frac{1+bc}{1+b} + \frac{1+ca}{1+c} = ab\frac{1+c}{1+a} + bc\frac{1+a}{1+b} + ca\frac{1+b}{1+c}$$
$$\geq 3\sqrt[3]{ab\frac{1+c}{1+a} \cdot bc\frac{1+a}{1+b} \cdot ca\frac{1+b}{1+c}} = 3\sqrt[3]{(abc)^2} = 3.$$

Example 8. Prove that for any $a, b, c > 0$ we have

$$\frac{a^3}{b} + \frac{b^3}{c} + \frac{c^3}{a} \geq ab + bc + ca.$$

Solution. By AM-GM inequality we have

$$\frac{a^3}{b} + \frac{b^3}{c} + bc \geq 3\sqrt[3]{\frac{a^3}{b} \cdot \frac{b^3}{c} \cdot (bc)} = 3ab.$$

Similarly,

$$\frac{b^3}{c} + \frac{c^3}{a} + ca \geq 3\sqrt[3]{\frac{b^3}{c} \cdot \frac{c^3}{a} \cdot (ca)} = 3bc$$

and

$$\frac{c^3}{a} + \frac{a^3}{b} + ab \geq 3\sqrt[3]{\frac{c^3}{a} \cdot \frac{a^3}{b} \cdot (ab)} = 3ca.$$

Adding the above inequalities, we obtain

$$2\left(\frac{a^3}{b} + \frac{b^3}{c} + \frac{c^3}{a}\right) + ab + bc + ca \geq 3(ab + bc + ca)$$

which proves our original inequality.

Example 9. Prove that if a and b are positive real numbers,

$$\sqrt[3]{\frac{a}{b}} + \sqrt[3]{\frac{b}{a}} \leq \sqrt[3]{2\,(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)}\,.$$

Solution. Recall that

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3.$$

Cubing both sides yields

$$\frac{a}{b}+3\left(\sqrt[3]{\frac{a}{b}}\right)^2\left(\sqrt[3]{\frac{b}{a}}\right)+3\left(\sqrt[3]{\frac{b}{a}}\right)^2\left(\sqrt[3]{\frac{a}{b}}\right)+\frac{b}{a} \leq 2\left(2+\frac{a}{b}+\frac{b}{a}\right)\,.$$

Simplifying this yields

$$3\sqrt[3]{\frac{a}{b}} + 3\sqrt[3]{\frac{b}{a}} \leq 4 + \frac{a}{b} + \frac{b}{a}\,. \tag{1}$$

Now by the AM-GM inequality,

$$1 + 1 + \frac{a}{b} \geq 3\sqrt[3]{\frac{a}{b}}$$

and

$$1 + 1 + \frac{b}{a} \geq 3\sqrt[3]{\frac{b}{a}}$$

with equality in both cases if and only if $a = b$. Adding these two inequalities together yields the required inequality (1).

Example 10. The positive real numbers a, b, c satisfy $a+b+c=1$.

Prove that

$$\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right) \geq 8.$$

Solution. Observe first that

$$\frac{1}{a}-1=\frac{1-a}{a}=\frac{(a+b+c)-a}{a}=\frac{b+c}{a} \geq \frac{2\sqrt{bc}}{a}.$$

Similarly,

$$\frac{1}{b}-1=\frac{1-b}{b}=\frac{(a+b+c)-b}{b}=\frac{c+a}{b} \geq \frac{2\sqrt{ca}}{b},$$

$$\frac{1}{c}-1=\frac{1-c}{c}=\frac{(a+b+c)-c}{c}=\frac{a+b}{c} \geq \frac{2\sqrt{ab}}{c}.$$

We multiply the above inequalitis and obtain

$$\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right) \geq 8\frac{\sqrt{(abc)^2}}{abc}=8.$$

Example 11. The positive real numbers a, b, c satisfy $a+b+c=1$. Prove that

$$\left(\frac{1}{a}+1\right)\left(\frac{1}{b}+1\right)\left(\frac{1}{c}+1\right) \geq 4^3.$$

Solution. First, by AM-GM inequality we find

$$1 = a+b+c \geq 3\sqrt[3]{abc}$$

so $abc \leq 127$. Now, we compute

$$\begin{aligned}\left(\frac{1}{a}+1\right)\left(\frac{1}{b}+1\right)\left(\frac{1}{c}+1\right) &= 1 + \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) + \left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right) + \frac{1}{abc} \\ &\geq 1 + 3\sqrt[3]{\frac{1}{abc}} + 3\sqrt[3]{\frac{1}{(abc)^2}} + \frac{1}{abc} \\ &\geq 1 + 3\sqrt[3]{27} + 3\sqrt[3]{27^2} + 27 \\ &= 64 = 4^3.\end{aligned}$$

Example 12. The positive real numbers a, b, c satisfy the double inequality

$$\frac{b^2}{a+b}+\frac{c^2}{b+c}+\frac{a^2}{c+a} \geq \frac{c^2}{a+b}+\frac{a^2}{b+c}+\frac{b^2}{c+a} \geq \frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a}.$$

Prove that $a = b = c$.

Solution. Looking at the first and the last term of our inequality, we observe that they are equal. Indeed,

$$\begin{aligned} &\left(\frac{b^2}{a+b}+\frac{c^2}{b+c}+\frac{a^2}{c+a}\right)-\left(\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a}\right) \\ &= \frac{b^2-a^2}{a+b}+\frac{c^2-b^2}{b+c}+\frac{a^2-c^2}{c+a} \\ &= (b-a)+(c-b)+(a-c)=0. \end{aligned}$$

It follows that

$$\frac{b^2}{a+b}+\frac{c^2}{b+c}+\frac{a^2}{c+a}=\frac{c^2}{a+b}+\frac{a^2}{b+c}+\frac{b^2}{c+a}$$

so

$$\frac{b^2-c^2}{a+b}+\frac{c^2-a^2}{b+c}+\frac{a^2-b^2}{c+a}=0.$$

Direct calculations show that this implies

$$a^2b^2+b^2c^2+c^2a^2-a^4-b^4-c^4=0$$

which can be rewritten into $(a^2-b^2)^2+(b^2-c^2)^2+(c^2-a^2)^2=0$. This easily yields $a=b=c$.