Circle and Cyclic Quadrilaterals

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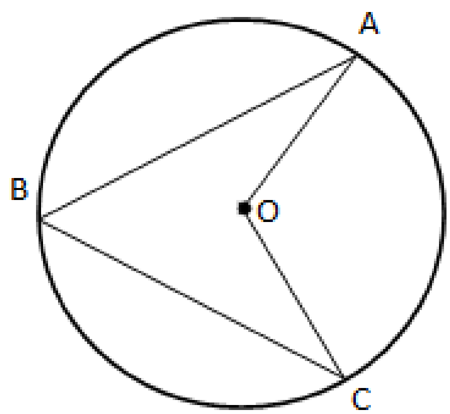
Basic Facts About Circles

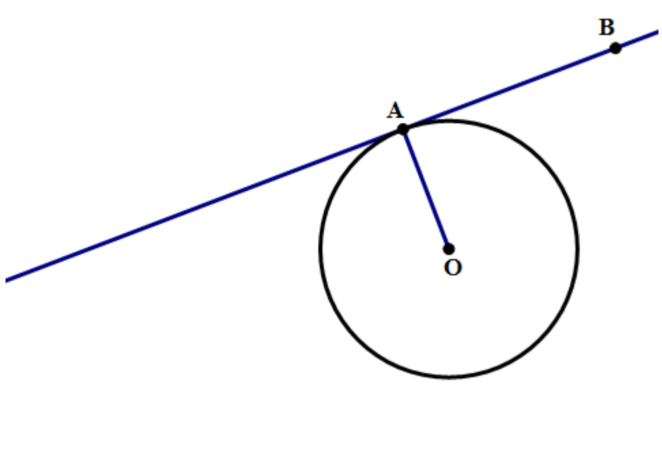
- A central angle is an angle whose vertex is at the center of the circle. Its measure equals the measure of the intercepted arc.
- An angle whose vertex lies on the circle and legs intersect the circle is called inscribed in the circle. Its measure equals half length of the subtended arc of the circle.



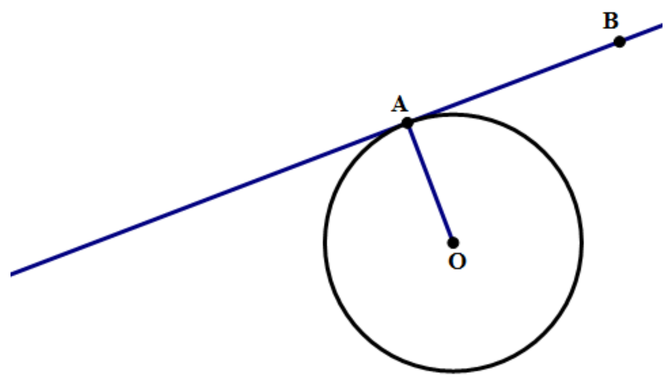
$\angle AOC=$ central angle, $\angle AOC = \widehat{AC}$

$\angle ABC=$inscribed angle, $\angle ABC = \frac{\widehat{AC}}{2}$

- A line that has exactly one common point with a circle is called tangent to the circle.

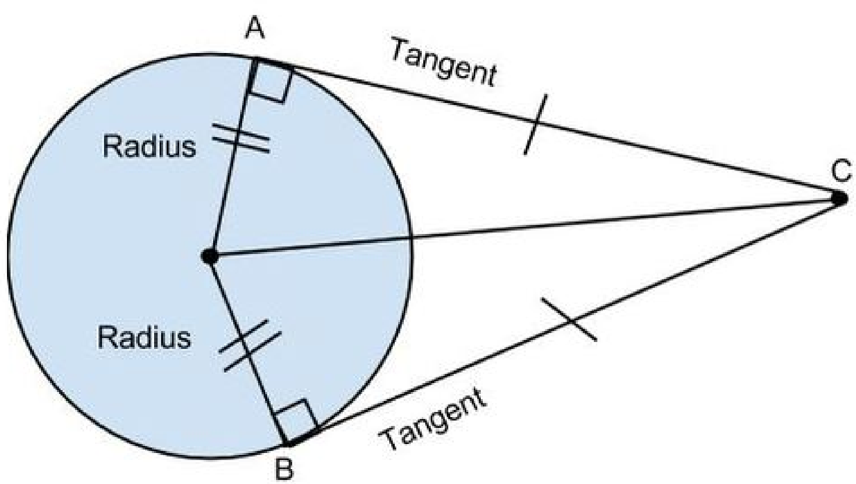


- The tangent at a point A on a circle is perpendicular to the diameter passing through A.



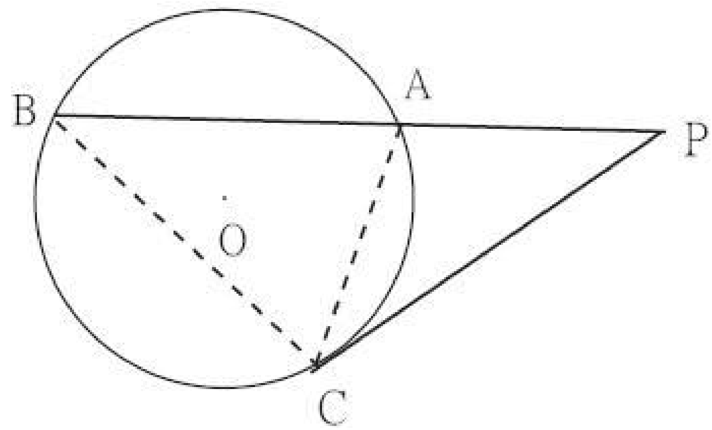
$$OA \perp AB$$

- Through a point A outside of a circle, exactly two tangent lines can be drawn. The two tangent segments drawn from an exterior point to a circle are equal.



$$OA = OB, \angle OBC = \angle OAC = 90^o \Longrightarrow \Delta OAB \equiv \Delta OBC$$

- The value of the angle between chord AB and the tangent line to the circle that passes through A equals half the length of the arc AB.



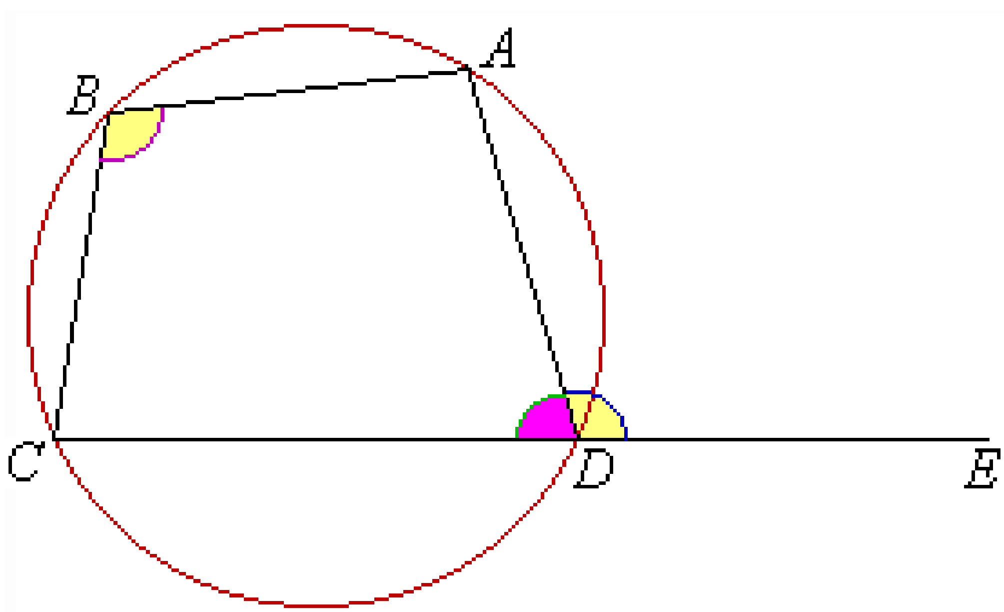
$$AC, BC = \text{ chords} \qquad CP = \text{ tangent}$$

$$\angle ABC = \frac{\widehat{AC}}{2}, \quad \angle ACP = \frac{\widehat{AC}}{2}$$

- The line passing through the centres of two tangent circles also contains their tangent point.

Cyclic Quadrilaterals

- A convex quadrilateral is called cyclic if its vertices lie on a circle.
- A convex quadrilateral is cyclic if and only if one of the following equivalent conditions hold:
  (1) The sum of two opposite angles is 180^o;
  (2) One angle formed by two consecutive sides of the quadrilateral equals the external angle formed by the other two sides of the quadrilateral;
  (3) The angle between one side and a diagonal equals the angle between the opposite side and the other diagonal.



Example 1. Two circles are internally tangent at K. Two lines passing through K intersect the two circles at A, C and B, D respectively. Prove that $AB||CD$.

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Solution. Denote by KL the common tangent line to the two circles. Observe that

$$\angle LKB = \angle KAB = \frac{\widehat{KB}}{2} \quad \text{and} \quad \angle LKB = \angle KCD = \frac{\widehat{KD}}{2}.$$

Hence, $\angle KAB = \angle KCD$ which shows that the lines AB and CD are parallel.

Example 2. Circles $\mathscr{C}_1$ and $\mathscr{C}_2$ having centres at O_1 and O_2 are externally tangent at P. Denote by AB and CD the two common tangents to these circles, so that A, D lie on $\mathscr{C}_1$ and B, C lie on $\mathscr{C}_2$. Show that

(a) $AD || BC$ (b) $AP \perp BP$.

Solution. (a) Observe first that the points O_1, O_2 and P are collinear. Then, triangles O_1AD and O_2BC are isosceles and $\angle AO_1D = \angle BO_2C$. It follows that $\angle O_1AD = \angle O_2BA$. This further yields $\angle O_1AB + \angle O_2BC = 180^0$, so $AD || BC$.

(b) Denote by Q the intersection between AB and the common tangent line to the two circles through P. Then, QA and QP and tangent to circle $\mathscr{C}_1$, so $QA = QP$. Similarly, QB and QP and tangent to circle $\mathscr{C}_2$, so $QB = QP$. Thus,

$$QA = QB = QP,$$

which means that triangles QAP and QBP are isosceles. Denote by x and y the measure of angles $\angle QAP$ and QBP respectively. It follows that $\angle APB = x + y$ and then, since

$$\angle PAB + \angle PBA + \angle APB = 180^0,$$

one gets $x + y = 90^0$, that is $\angle APB = 90^0$, so $AP \perp BP$.

Example 3. Let BD and CE be altitudes in a triangle ABC. Prove that if $DE||BC$, then $AB = AC$.

Solution. Let us observe first that $\angle BEC = \angle CDE = 90^o$, so $BCDE$ is cyclic. It follows that $\angle AED = \angle ACB$ (1)
On the other hand, $DE||BC$ implies $\angle AED = ABC$ (2)
From (1) and (2) it follows that $\angle ABC = \angle ACB$ so ΔABC is isosceles.

Example 4. In the cyclic quadrilateral $ABCD$, the perpendicular from B on AB meets DC at B' and the perpendicular from D on DC meets AB at D'. Prove that $B'D'||AC$.

Solution. Since $ABCD$ is cyclic we have $\angle ACD = \angle ABD$. Similarly, $BD'DB'$ is cyclic (because $\angle B'DD' + \angle B'BD' = 180^o$) implies $\angle DB'D' = \angle D'BD$. Hence $\angle DCA = \angle CB'D'$, so that $AC||B'D'$.

Example 5. A line parallel to the base BC of triangle ABC intersects AB and AC at P and Q respectively. The circle passing through P and tangent to AC at Q intersects AB again at R. Prove that $BCQR$ is cyclic.

Solution. It is enough to prove that $\angle ARQ = \angle ACB$.
Indeed, since ΔPRQ is inscribed in the circle $\Longrightarrow \angle PRQ = \frac{\widehat{PQ}}{2}$.
Since AC is tangent to the circle passing through $P, Q, R \Longrightarrow \angle AQP = \frac{\widehat{PQ}}{2}$.
Hence, $\angle PRQ = \angle AQP$. Now, since $PQ||BC$ it follows that $\angle AQP = \angle ACB$. Thus, $\angle ARQ = \angle ACB$ which shows that $BCQR$ is cyclic.

Example 6. The diagonals of the cyclic quadrilateral $ABCD$ are perpendicular and meet at P. The perpendicular from P to AD meets BC at Q. Prove that $BQ = CQ$.

Solution. Denote by M the intersection between AD and PQ.

$$\left.\begin{array}{ll} \angle MPD = \angle BPQ & \text{(opposite angles)} \\ \angle MPD = \angle MAP & (= 90^o - \angle APM) \\ \angle MAP = \angle CBP & (ABCD \text{ cyclic}) \end{array}\right| \Longrightarrow \angle BPQ = \angle CBP$$

Hence, ΔQBP is isosceles which further yields $BQ = QP$ (1)

Similarly we have

$$\left.\begin{array}{ll} \angle APM = \angle CPQ & \text{(opposite angles)} \\ \angle APM = \angle ADP & (= 90^o - \angle MPD) \\ \angle ADP = \angle QCP & (ABCD \text{ cyclic}) \end{array}\right| \Longrightarrow \angle CPQ = \angle QCP$$

Hence, ΔQCP is isosceles which further yields $CQ = QP$ (2)

From (1) and (2) it follows that $BQ = CQ$.

Example 7. Let E and F be two points on the sides BC and DC of the square $ABCD$ such that $\angle EAF = 45^o$. Let M and N be the intersection of the diagonal BD with AE and AF respectively. Let P be the intersection of MF and NE. Prove that $AP \perp EF$.

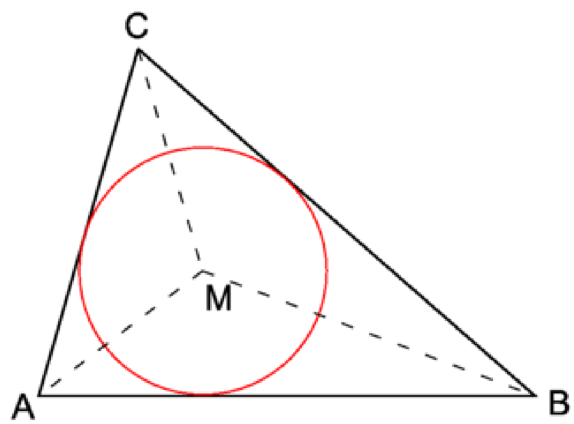
Solution. $\angle EAN = \angle EBN = 45^o$ so $ABEN$ is cyclic. It follows that $\angle ANE = 180^o - \angle ABE = 90^0$, so $NE \perp AF$.
Similarly, $ADFM$ is cyclic so $\angle AMF = 180^o - \angle ADF = 90^o$ which yields $AE \perp FM$. It follows that EN and FM are altitudes in ΔAEF, so P is the orthocentre of ΔAEF. This implies $AP \perp EF$.

Example 8. (Japan Maths Olympiad) Let $ABCD$ be a cyclic quadrilateral. Prove that the incentres of triangles ABC, BCD, CDA, ADB are the vertices of a rectangle.

Note. The incenter is the intersection of angles' bisectors.

Solution. We shall start with the following auxiliary result.

Lemma. If M is the incentre of ΔABC then $\angle AMB = 90^o + \frac{\angle ACB}{2}$.



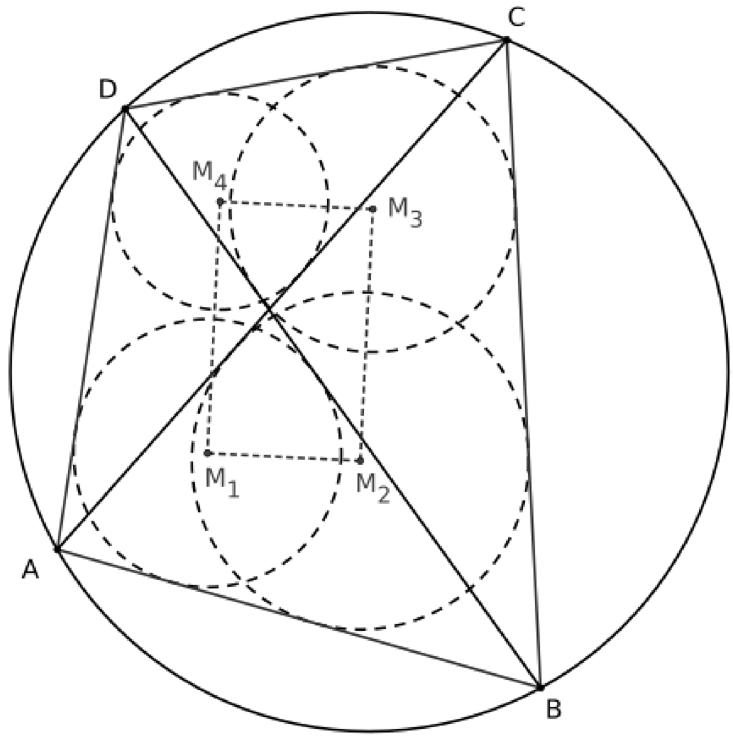
Proof of Lemma. In ΔBMC we have

$$\begin{aligned}\angle AMB &= 180^o - \angle MAB - \angle MBA \\ &= 180^0 - \frac{\angle BAC}{2} - \frac{\angle ABC}{2} \\ &= 180^0 - \frac{\angle BAC + \angle ABC}{2} \\ &= 180^0 - \frac{180^o - \angle ACB}{2} \\ &= 90^0 + \frac{\angle ACB}{2}.\end{aligned}$$

Returning to our solution, denote by M_1, M_2, M_3, M_4 the incentres of triangles DAB, ABC, BCD and CDA respectively.

M_1 is the incentre of $\Delta DAB \Longrightarrow \angle AM_1B = 90^0 + \frac{\angle ADB}{2}$. (1)

M_2 is the incentre of $\Delta ABC \Longrightarrow \angle AM_2B = 90^0 + \frac{\angle ACB}{2}$. (2)



$ABCD$ is cyclic $\Longrightarrow \angle ACB = \angle ADB$.   (3)

Combining (1), (2) and (3) we find $\angle AM_1B = \angle AM_2B$ so ABM_2M_1 is cyclic. It follows that

$$\angle BM_2M_1 = 180^0 - \angle BAM_1 = 180^0 - \frac{\angle BAD}{2}. \text{ (4)}$$

Similarly BCM_3M_1 is cyclic so

$$\angle BM_2M_3 = 180^0 - \angle BCM_3 = 180^0 - \frac{BCD}{2}. \text{ (5)}$$

From (4) and (5) we now deduce

$$\angle M_1M_2M_3 = 360^0 - (\angle BM_2M_1 + \angle BM_2M_3) = \frac{\angle BAD}{2} + \frac{BCD}{2} = 90^0.$$

In the same way we obtain that all angles of the quadrilateral $M_1M_2M_3M_4$ have measure 90^0 and this finishes our proof.

Practice Problems

(1) Let A', B' and C' be points on the sides BC, CA and AB of triangle ABC. Prove that the circumcentres of triangles $AB'C'$, $BA'C'$ and $CA'B'$ have a common point.

Hint: Denote by M the point of intersection of circumcentres of triangles $AB'C'$ and $BA'C'$. Prove that $MA'CB'$ is cyclic so the circumcentre of triangle $A'CB'$ passes through M as well.

(2) In the convex quadrilateral $ABCD$ the diagonals AC and BD are perpendicular and meet at O. Prove that the projections of O on the quadrilateral sides are the vertices of a cyclic quadrilateral.

(3) (Simpson's line) Let M be a point on the circumcircle of triangle ABC. Prove that the feet of perpendiculars from M to the sides AB, BC and CA are collinear.