Circle and Cyclic Quadrilaterals

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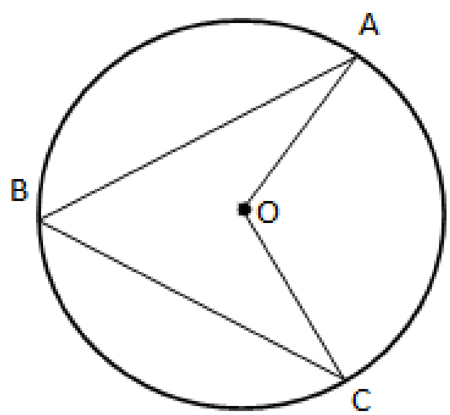
Basic Facts About Circles

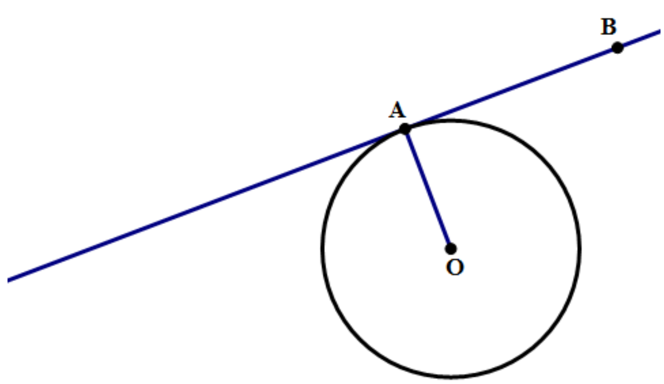
- A central angle is an angle whose vertex is at the center of the circle. It measure is equal the measure of the intercepted arc.
- An angle whose vertex lies on the circle and legs intersect the cirlc is called inscribed in the circle. Its measure equals half length of the subtended arc of the circle.



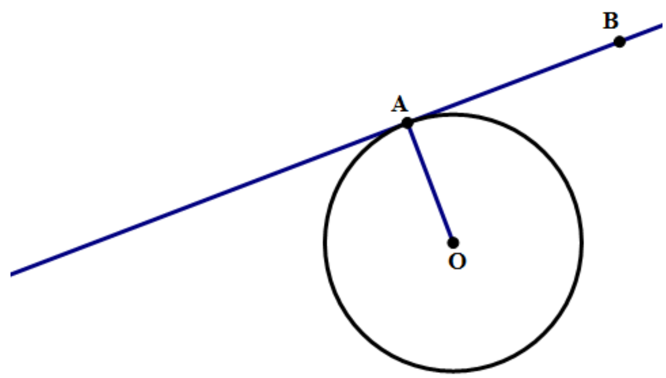
$\angle AOC$= contral angle, $\angle AOC = \widehat{AC}$

$\angle ABC$=inscribed angle, $\angle ABC = \frac{\widehat{AC}}{2}$

- A line that has exactly one common point with a circle is called tangent to the circle.

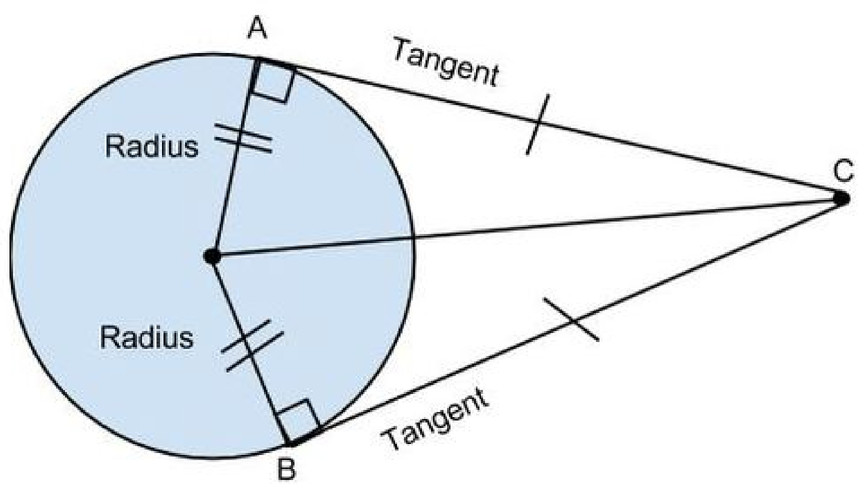


- The tangent at a point A on a circle of is perpendicular to the diameter passing through A.



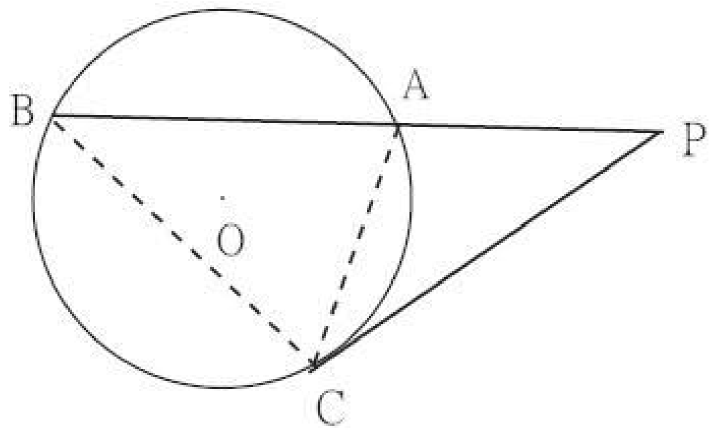
$$OA \perp AB$$

- Through a point A outside of a circle, exactly two tangent lines can be drawn. The two tangent segments drawn from an exterior point to a cricle are equal.



$$OA = OB,\ \angle OBC = \angle OAC = 90^o \Longrightarrow \Delta OAB \equiv \Delta OBC$$

- The value of the angle between chord AB and the tangent line to the circle that passes through A equals half the length of the arc AB.



$$AC, BC = \text{ chords} \qquad CP = \text{ tangent}$$

$$\angle ABC = \frac{\widehat{AC}}{2}, \quad \angle ACP = \frac{\widehat{AC}}{2}$$

- The line passing through the centres of two tangent circles also contains their tangent point.

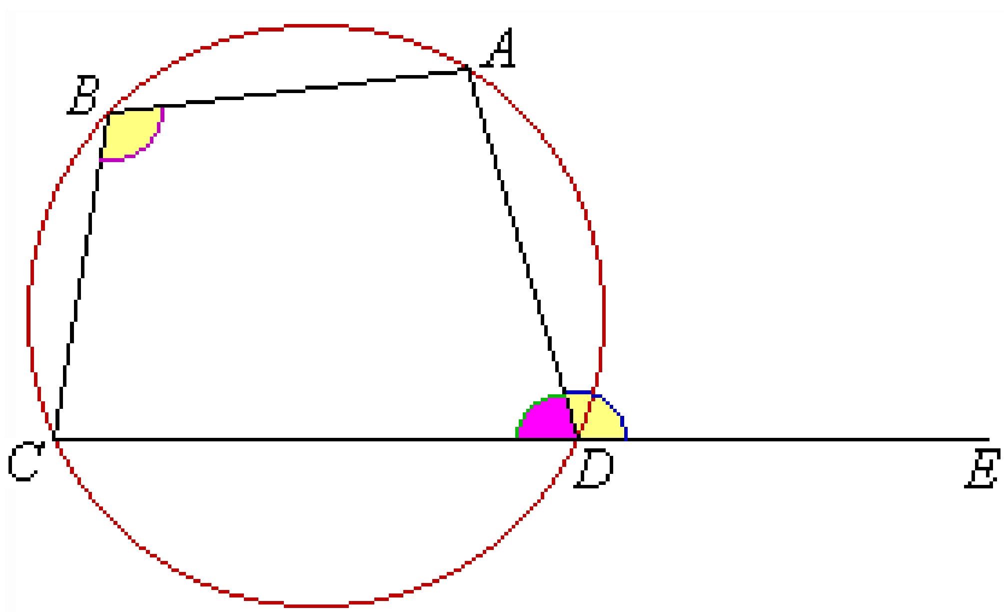
Cyclic Quadrilaterals

- A convex quadrilateral is called cyclic if its vertices lie on a circle.
- A convex quadrilateral is cyclic if and only if one of the following equivalent conditions hold:

(1) The sum of two opposite angles is 180^o;

(2) One angle formed by two consecutive sides of the quadrilateral equal the external angle formed by the other two sides of the quadrilateral;

(3) The angle between one side and a diagonal equals the angle between the opposite side and the other diagonal.



Example 1. Let BD and CE be altitudes in a triangle ABC. Prove that if $DE||BC$, then $AB = AC$.

Solution. Let us observe first that $\angle BEC = \angle CDE = 90^o$, so $BCDE$ is cyclic. It follows that $\angle AED = \angle ACB \quad (1)$
On the other hand, $DE||BC$ implies $\angle AED = ABC \quad (2)$
From (1) and (2) it follows that $\angle ABC = \angle ACB$ so ΔABC is isosceles.

Example 2. In the cyclic quadrilateral $ABCD$, the perpendicular from B on AB meets DC at B' and the perpendicular from D on DC meets AB at D'. Prove that $B'D'||AC$.

Solution. Since $ABCD$ is cyclic we have $\angle ACD = \angle ABD$. Similarly, $BD'DB'$ is cyclic (because $\angle B'DD' + \angle B'BD' = 180^o$) implies $\angle DB'D' = \angle D'BD$. Hence $\angle DCA = \angle CB'D'$, so that $AC||B'D'$.

Example 3. A line parallel to the base BC of triangle ABC intersects AB and AC at P and Q respectively. The circle passing through P and tangent to AC at Q intersects AB again at R. Prove that $BCQR$ is cyclic.

Solution. It is enough to proce that $\angle ARQ = \angle ACB$.
Indeed, since ΔPRQ is inscribed in the circle $\Longrightarrow \angle PRQ = \frac{\widehat{PQ}}{2}$.
Since AC is tangent to the circle passing through $P, Q, R \Longrightarrow \angle AQP = \frac{\widehat{PQ}}{2}$.
Hence, $\angle PRQ = \angle AQP$. Now, since $PQ || BC$ it follows that $\angle AQP = \angle ACB$. Thus, $\angle ARQ = \angle ACB$ which shows that $BCQR$ is cyclic.

Example 4. The diagonals of the cyclic quadrilateral $ABCD$ are perpendicular and meet at P. The perpendicular from P to AD meets BC at Q. Prove that $BQ = CQ$.

Solution. Denote by M the intersection between AD and PQ.

$$\left.\begin{array}{ll} \angle MPD = \angle BPQ & \text{(opposite angles)} \\ \angle MPD = \angle MAP & (= 90^o - \angle APM) \\ \angle MAP = \angle CBP & (ABCD \text{ cyclic}) \end{array}\right| \Longrightarrow \angle BPQ = \angle CBP$$

Hence, δQBP is isosceles which further yields $BQ = QP$ (1)

Similarly we have

$$\left.\begin{array}{ll} \angle APM = \angle CPQ & \text{(opposite angles)} \\ \angle APM = \angle ADP & (= 90^o - \angle MPD) \\ \angle ADP = \angle QCP & (ABCD \text{ cyclic}) \end{array}\right| \Longrightarrow \angle CPQ = \angle QCP$$

Hence, δQCP is isosceles which further yields $CQ = QP$ (2)

From (1) and (2) it follows that $BQ = CQ$.

Example 5. Let E and F be two points on the sides BC and DC of the square $ABCD$ such that $\angle EAF = 45^o$. Let M and N be the intersection of the diagonal BD with AE and AF respetively. Let P be the intersection of MF and NE. Prove that $AP \perp EF$.

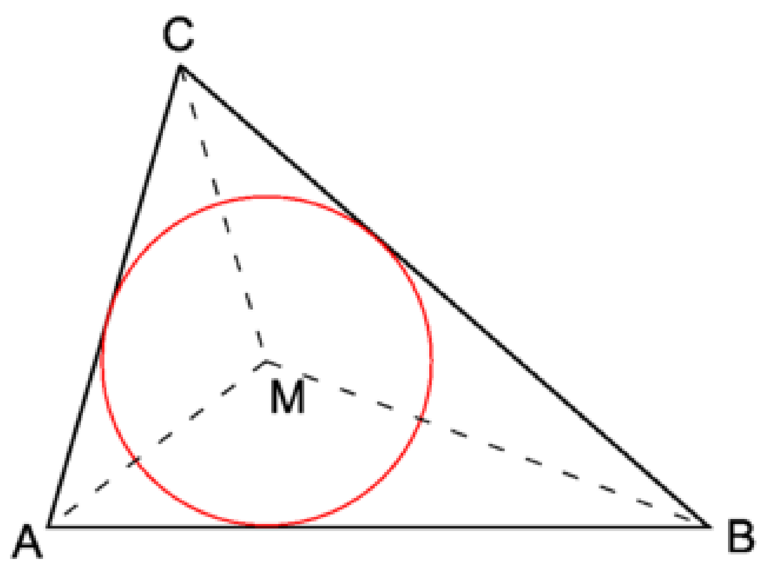
Solution. $\angle EAN = \angle EBN = 45^o$ so $ABEN$ is cyclic. It follows that $\angle ANE = 180^o - \angle ABE = 90^0$, so $NE \perp AF$.
Similarly, $ADFM$ is cyclic so $\angle AMF = 180^o - \angle ADF = 90^o$ which yields $AE \perp FM$. It follows that EN and FM are altitudes in ΔAEF, so P is the orthocentre of ΔAEF. This imples $AP \perp EF$.

Example 6. Let $ABCD$ be a cyclic quadrilateral. Prove that the incentres of traingles ABC, BCD, CDA, ADB are the vertices of a rectangle.

Note. The incenter is the intersection of angles' bisectors.

Solution. We shall start with the following auxiliary result.

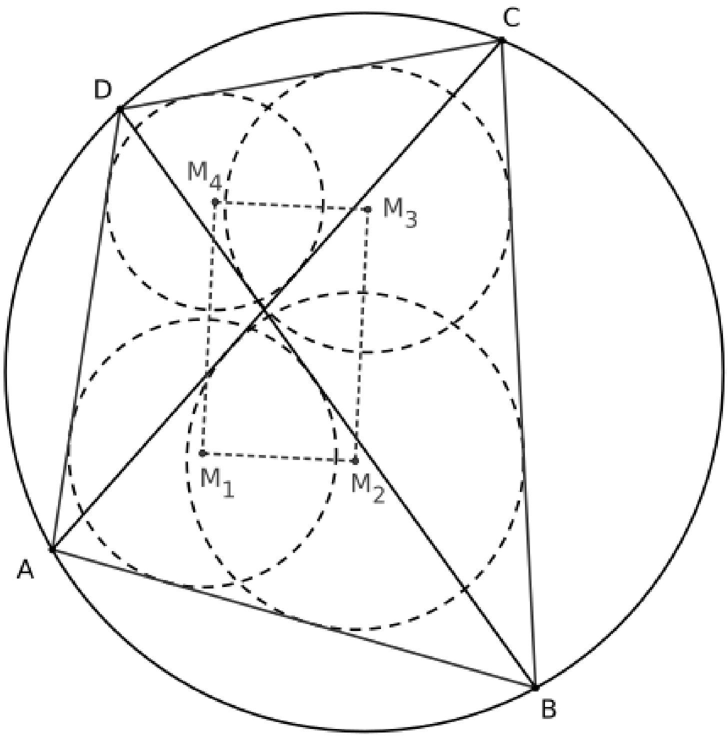
Lemma. If M is the incentre of ΔABC then $\angle AMB = 90^o + \frac{\angle ACB}{2}$.



Proof of Lemma. In ΔBMC we have

$$\begin{aligned}\angle AMB &= 180^o - \angle MAB - \angle MBA \\ &= 180^o - \frac{\angle BAC}{2} - \frac{\angle ABC}{2} \\ &= 180^o - \frac{\angle BAC + \angle ABC}{2} \\ &= 180^o - \frac{180^o - \angle ACB}{2} \\ &= 90^o + \frac{\angle ACB}{2}.\end{aligned}$$

Returning to our solution, denote by M_1, M_2, M_3, M_4 the incentres of traingles DAB, ABC, BCD and CDA respectively.



M_1 is the incentre of $\Delta DAB \Longrightarrow \angle AM_1B = 90^o + \frac{\angle ADB}{2}$. (1)
M_2 is the incentre of $\Delta ABC \Longrightarrow \angle AM_2B = 90^o + \frac{\angle ACB}{2}$. (2)
$ABCD$ is cyclic $\Longrightarrow \angle ACB = \angle ADB$. (3)
Combining (1), (2) and (3) we find $\angle AM_1B = \angle AM_2B$ so ABM_2M_1 is cyclic. It follows that

$$\angle BM_2M_1 = 180^o - \angle BAM_1 = 180^o - \frac{\angle BAD}{2}. \text{ (4)}$$

Similarly BCM_3M_1 is cyclic so

$$\angle BM_2M_3 = 180^o - \angle BCM_3 = 180^o - \frac{BCD}{2}. \text{ (5)}$$

From (4) and (5) we now deduce

$$\angle M_1M_2M_3 = 360^o - (\angle BM_2M_1 + \angle BM_2M_3) = \frac{\angle BAD}{2} + \frac{BCD}{2} = 90^o.$$

In the same way we obtain that all angles of the quadrilateral $M_1M_2M_3M_4$ have meaure 90^o and this finishes our proof.

Example 7. Let A', B' and C' be points on the sides BC, CA and AB of trinagle ABC. Prove that the circumcentres of traingles $AB'C'$, $BA'C'$ and $CA'B'$ have a common point.

Solution. Denote by M the point of intersection of circumcentres of triangl;es $AB'C'$ and $BA'C'$. We prove that $MA'CB'$ ic cyclic so the circumcentre of triangle $A'CB'$ opasses through M as well.