# UCD Mathematics Enrichment Programme

## Solutions to Selection Test, 14 February 2015

1. (a) Which is the larger number: $A = 200!$ or $B = 100^{200}$? Justify your answer.
   (b) Which is the larger number: $A = 2000!$ or $B = 100^{2000}$? Justify your answer.

**Solution:**

(a) $B > A$:
We have

$$\begin{aligned} \frac{A}{B} &= \frac{200}{100} \cdot \frac{199}{100} \cdots \frac{100}{100} \cdot \frac{99}{100} \cdots \frac{1}{100} \\ &= \frac{200}{100} \left( \frac{199}{100} \cdot \frac{1}{100} \right) \cdot \left( \frac{198}{100} \cdot \frac{2}{100} \right) \cdots \left( \frac{101}{100} \cdot \frac{99}{100} \right) \end{aligned}$$

This expression is $< 1$ since the product of the first three terms is less than 1 and each of the products

$$\left( \frac{100+k}{100} \cdot \frac{100-k}{100} \right) = \left( 1 + \frac{k}{100} \right) \cdot \left( 1 - \frac{k}{100} \right) = 1 - \frac{k^2}{100^2}$$

is less than 1. So $A/B < 1$ as claimed.

(b) $A > B$:
The product $A/B$ is the product of all terms $n/100$ as $n$ goes from 1 to 2000. This product contains the 99 products of three terms

$$\left( \frac{k}{100} \cdot \frac{2000-k}{100} \cdot \frac{1000-k}{100} \right)$$

as $k$ goes from 1 to 99. Each of these terms is $> 1$ since each is greater than

$$\left( \frac{1}{100} \cdot \frac{1900}{100} \cdot \frac{900}{100} \right) = \frac{19 \cdot 9 \cdot 100^2}{100^3} > 1.$$

All remaining factors in the product $A/B$ are of the form $n/100$ with $n \geq 100$. Thus $A/B > 1$.

2. Show that for all positive integers $n \geq 2$ we have

$$n! < \left( \frac{n+1}{2} \right)^n.$$

**Solution:** The AM-GM inequality gives

$$1 + 2 + \cdots + n > n \sqrt[n]{1 \cdot 2 \cdots n} = n \sqrt[n]{n!}.$$

Therefore

$$\frac{n(n+1)}{2} = 1 + 2 + \cdots + n > n \sqrt[n]{n!}.$$

Cancelling $n$ on both sides gives

$$\frac{n+1}{2} > \sqrt[n]{n!}$$

and taking $n$-th powers gives the required inequality.

3. In triangle $ABC$ we denote by $A'$, $B'$, $C'$ the midpoints of sides $BC$, $CA$ and $AB$ respectively. We extend $AA'$ beyond $A'$ with $A'M = AA'$. We extend $BB'$ beyond $B'$ with $B'N = BB'$ and extend $CC'$ beyond $C'$ with $C'P = CC'$. Denote by $G_1$, $G_2$ and $G_3$ the centroids of triangles $MBC$, $NAC$ and $PAB$. Prove that triangles $ABC$ and $G_1G_2G_3$ have the same area.

   **Solution:** We note first that $ABMC$ is a parallelogram since its diagonals $AM$ and $BC$ have the same midpoint $A'$. Similarly $ABCN$ and $ACBP$ are parallelograms. This yields $AN||BC$ and $AP||BC$ so $A, N, P$ are collinear. Similarly $B, M, P$ and $C, M, N$ are collinear. It follows that $A, B, C$ are the midpoints of $NP$, $MP$ and $MN$ respectively. Denote by $G$ the centroid of $ABC$. Using the property of the centroid that it is located at 2/3 of the vertex and 1/3 of the base, we deduce $AG = GG_1 = G_1M$. Thus, $G_1$ is the midpoint of $GM$. Similarly, $G_2$ is the midpoint of $GN$ and $G_3$ is the midpoint of $GP$. Hence $G_1G_2 = \frac{MN}{2}$, $G_2G_3 = \frac{NP}{2}$, $G_1G_3 = \frac{MP}{2}$. This implies $[G_1G_2G_3] = \frac{1}{4}[MNP] = [ABC]$.

4. Let $x, y, z, w$ be positive real numbers, and suppose that $xyzw = 16$. Show that

$$\frac{x^2}{x+y} + \frac{y^2}{y+z} + \frac{z^2}{z+w} + \frac{w^2}{w+x} \geq 4$$

   with equality only when $x = y = z = w = 2$.

   **Solution:** The Cauchy inequality gives

$$\left(\left(\frac{x}{\sqrt{x+y}}\right)^2 + \left(\frac{y}{\sqrt{y+z}}\right)^2 + \left(\frac{z}{\sqrt{z+w}}\right)^2 + \left(\frac{w}{\sqrt{w+x}}\right)^2\right) \times \left((\sqrt{x+y})^2 + (\sqrt{y+z})^2 + (\sqrt{z+w})^2 + (\sqrt{w+x})^2\right) \geq (x+y+z+w)^2,$$

   with equality only when $x = y = z = w$. This simplifies to:

$$\left(\frac{x^2}{x+y} + \frac{y^2}{y+z} + \frac{z^2}{z+w} + \frac{w^2}{w+x}\right) \cdot 2(x+y+z+w) \geq (x+y+z+w)^2$$

   and hence

$$\left(\frac{x^2}{x+y} + \frac{y^2}{y+z} + \frac{z^2}{z+w} + \frac{w^2}{w+x}\right) \geq \frac{x+y+z+w}{2}.$$

   Applying the AM-GM to the right-hand term gives

$$\left(\frac{x^2}{x+y} + \frac{y^2}{y+z} + \frac{z^2}{z+w} + \frac{w^2}{w+x}\right) \geq 2\sqrt[4]{xyzw}$$

   with equality only when $x = y = z = w$. Since $xyzw = 16$, the result follows at once.

5. For any positive integer $k$ define

$$H_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}.$$

   Prove that for $n \geq 1$,

$$1 + \frac{1}{n+1}(H_1 + H_2 + \cdots + H_n) = H_{n+1}.$$

   **Solution.** For $n = 1$ we have $1 + \frac{1}{2}H_1 = 1 + \frac{1}{2} = H_2$. Assume that the relation holds for some $n \geq 1$, i.e.,

$$1 + \frac{1}{n+1}\sum_{k=1}^{n} H_k = H_{n+1}.$$

Then

$$\begin{aligned} 1+\frac{1}{n+2}\sum_{k=1}^{n+1} H_k &= 1+\frac{1}{n+2}H_{n+1}+\frac{n+1}{n+2}\cdot\frac{1}{n+1}\sum_{k=1}^{n} H_k \\ &= 1+\frac{1}{n+2}H_{n+1}-\frac{n+1}{n+2}+\frac{n+1}{n+2}H_{n+1} \\ &= H_{n+1}+\frac{1}{n+2} \\ &= H_{n+2}\ , \end{aligned}$$

and the relation holds also for $n+1$. Therefore, by the principle of induction, the relation holds for all $n \geq 1$.

6. We have a deck of $10,000$ cards, numbered from 1 to $10,000$. A step consists of removing every card which has a perfect square on it, and then renumbering the remaining cards, starting from 1, in a consecutive way (i.e., numbering them 1, 2, 3, etc.)

   Find, with proof, the number of steps needed to remove all but one card.

   **Solution.** On the first step, we remove the cards numbered $1^2$, $2^2$, $3^2$, $\ldots$, $100^2$. Then 9900 cards remain. Since $99^2 < 9900 < 100^2$, we remove the cards numbered $1^2$, $2^2$, $3^2$, $\ldots$, $99^2$ in the second step. After that, $9900 - 99 = 9801 = 99^2$ cards are left, which is a perfect square.

   In general, if we start with $n^2$ cards for any $n \geq 2$, we remove $n$ cards in the first step, after which $n^2 - n$ cards remain. Since $(n-1)^2 = n^2 - 2n + 1 < n^2 - n < n^2$, we remove $n - 1$ cards in the second step. Then exactly $(n^2 - n) - (n - 1) = (n - 1)^2$ cards are left. So in two steps we reduce the number of cards from $n^2$ to $(n - 1)^2$.

   It follows that we need 99 pairs of steps, or $2 \cdot 99 = 198$ steps in total to remove all but one card.

7. Determine all triples $(a, b, c)$ of positive integers satisfying both of the following properties:

   (i) We have $a < b < c$, and $a$, $b$ and $c$ are three consecutive odd integers;

   (ii) The number $a^2 + b^2 + c^2$ consists of four equal digits.

   **Solution.** Since $a$, $b$ and $c$ are three consecutive odd positive integers, we can write $a = 2n - 1$, $b = 2n + 1$ and $c = 2n + 3$, where $n$ is a positive integer. Then

$$\begin{aligned} a^2+b^2+c^2 &= (2n-1)^2+(2n+1)^2+(2n+3)^2 \\ &= (4n^2-4n+1)+(4n^2+4n+1)+(4n^2+12n+9) \\ &= 12n^2+12n+11\ . \end{aligned}$$

   This needs to be a 4-digit number each of whose digits is equal to $p$, where $p \in \{0, 1, 2, \ldots, 9\}$. Hence the integer $12n^2 + 12n$ consists of four digits, of which the first two are equal to $p$ and the last two are equal to $p - 1$. Since $12n^2 + 12n$ is divisible by 2, $p - 1$ has to be even. So we have the following possibilities for $12n^2 + 12n$: 1100, 3322, 5544, 7766, and 9988. This integer must also be divisible by 3, so the only integer remaining is 5544: therefore $n^2 + n = 5544/12 = 462$. We can rewrite this as $n^2+n-462 = 0$. Factorizing this quadratic equation then gives $(n-21)(n+22) = 0$. Since $n$ is a positive integer, the only solution is $n = 21$. So the only triple satisfying the given properties is $(a, b, c) = (41, 43, 45)$.

8. (a) Find with proof all integers $x, y$ such that

$$\frac{x^4 + x^2y^2 + y^4}{3}$$

   is a prime number.

(b) Prove that if $x$ and $y$ are integers, then

$$\frac{x^4 + x^2y^2 + y^4}{5}$$

is not a prime number.

**Solution**: (a) If $x$ or $y$ is zero, there are clearly no solutions. Also, for any solution $(x, y)$, $(\pm x, \pm y)$ are also solutions.

Suppose that $(x, y)$ is a solution with $x, y$ positive integers. Since

$$x^4 + x^2y^2 + y^4 = (x^2 + xy + y^2)(x^2 - xy + y^2),$$

if $y = 1, x^2 - x + 1 = 1$ or 3, yielding $x = 1$ or $x = 2$. However, $x = y = 1$ does not yield a solution, so $y = 1$ implies $x = 2$ and $x^4 + x^2y^2 + y^4 = 3 \times 7$, giving a solution. If $y > 1$, then $x^2 \pm xy + y^2 = (x \pm \frac{y}{2})^2 + \frac{3y^2}{4} > 3$ unless $y = 2$ and $x = 1$, so, we must have $y = 2$ and $x = 1$ for a solution.

So the only solutions are: $(x, y) = (2, 1), (2, -1), (-2, 1), (-2, -1), (1, 2), (1, -2), (-1, 2)$ and $(-1, -2)$.

(b) Assume, for the sake of contradiction, that the result is false and that $x$ and $y$ are integers with

$$\frac{x^4 + x^2y^2 + y^4}{5}$$

prime. Arguing as in part (a), we see that $x$ and $y$ are nonzero and that we may assume both $x$ and $y$ are positive and $(x, y) \neq (1, 1)$. Since

$$x^4 + x^2y^2 + y^4 = (x^2 + xy + y^2)(x^2 - xy + y^2),$$

and $(x^2 + xy + y^2) > (x^2 - xy + y^2)$, we must have $(x^2 - xy + y^2) = 1$ or 5.

We deduce that, in the first case, $x = y = 1$ which does not satisfy our conditions. Hence we must have

$$x^2 - xy + y^2 = 5,$$

that is,

$$(x - \frac{y}{2})^2 + \frac{3y^2}{4} = 5.$$

Since $(x - \frac{y}{2})^2 \geq 0$, and $\frac{3y^2}{4} \geq \frac{27}{4} > 5$, for $y \geq 3$, we must have $y \leq 2$, and similarly $x \leq 2$. Also, at least one of $x, y$ must be odd. Suppose $y$ is odd. Then $y = 1$ and $x^2 - x + 1 = 5$, so $x^2 - x - 4 = 0$, so $4x^2 - 4x - 16 = 0$ and thus $(2x - 1)^2 = 17$, which is impossible since $\sqrt{17}$ is not an integer.A similar contradiction occurs if we assume that $x$ is odd.

So we have reached a contradiction. This proves the result.

9. A triangle has angles of $36°, 72°$ and $72°$. Prove that it has at least one side whose length is not an integer.

   **Solution:** Suppose for the sake of contradiction that the lengths of the sides are integers $a, b, b$ with the side of length $a$ opposite the angle $36°$. Applying the cosine rule we obtain

$$a^2 = b^2 + b^2 - 2b^2 \cos(36°).$$

   This implies that $\cos(36°)$ is a rational number. We now show that this is not true.

   Let $y = \cos(36°)$. Then, since $3\times$ $36° = 180° - 2 \times 36°$, we have

$$4y^3 - 3y = \cos(3 \times 36°) = -\cos(2 \times 36°) = -2y^2 + 1,$$

so

$$4y^3 + 2y^2 - 3y - 1 = 0,$$

that is,

$$(y+1)(4y^2 + 2y - 1) = 0.$$

Since $y \neq -1, 4y^2 + 2y - 1 = 0$ and $y = \frac{-1 \pm \sqrt{5}}{4}$, so $y = \frac{-1+\sqrt{5}}{4}$, as $y > 0$.

Since $\sqrt{5}$ is irrational, so is $y$. So our initial supposition led to a contradiction, and the result is proved.

10. Find with proof all positive integers $k$ such that, for $n = 2^k$, every prime number which divides $n! + 1$ also divides $n + 1$.

**Solution**: Suppose that $k > 2$ is a positive integer with the given property. Let $p$ be a prime dividing $n+1$ which also divides $n!+1$. Then $p$ does not divide $n!$, so $p > n$ and therefore $p = n+1$ and $n + 1$ is prime. Also, $n! + 1$ must divide a power of $p$, so, since $p$ is prime, $n! + 1 = p^m$, for some integer $m \geq 1$.

Since $n = 2^k > 4, p - 1 = 2^k$, and all the numbers $2, 4, 6, \;\ldots\;, 2^k - 2$ are even and $4 = 2^2$, so $n! + 1$ is divisible by $2^{r+k}$, where $r \geq (\frac{1}{2})(2^k - 2) + 1 = \frac{n}{2}$. ... (1)

Let $m = 2^s q$, where $s$ and $q$ are nonnegative integers with $q$ odd. Then, using the binomial theorem,

$$\begin{aligned} p^m &= (1 + 2^k)^m \\ & 1 + 2^k m + 2^{2k}\binom{m}{2} + 2^{3k}\binom{m}{3} + \;\ldots\; + \\ & + 2^{2jk}\binom{m}{j} + \;\ldots\; + 2^{mk}\binom{m}{m}. \quad \ldots \quad (2) \end{aligned}$$

Observe that

$$\binom{m}{2} = \frac{m(m-1)}{2} = \frac{2^s q(2^s q - 1)}{2}$$

is divisible by $2^{s-1}$(if $s > 0$) and thus $2^{2k}\binom{m}{2}$ is divisible by $2^{k+s+1}$, since $k > 1$. More generally,

$$\begin{aligned} \binom{m}{j} &= \frac{m(m-1)(m-2) \;\ldots\; (m-j+1)}{j!} \\ &= \frac{m}{j}\binom{m-1}{j-1} \end{aligned}$$

is divisible by $2^{s-j+1}$ (if $s \geq j$), and thus $2^{jk}\binom{m}{j}$ is divisible by $2^{k+s+1}$, for $2 \leq j \leq k$.

Hence equation (2) implies that

$$p^m = 1 + 2^{k+s}q + 2^{k+s+1}h,$$

for some integer $h$. Thus $2^{k+s}$ divides $p^m - 1$ and $2^{k+s+1}$ does not divide $p^m - 1$.

So, by (1), $k + s \geq k + r \geq k + \frac{n}{2}$, so $s \geq \frac{n}{2}$ and $2^s \geq 2^{n/2}$.

However, it is easy to prove by induction that $2^{n/2} \geq n$, for $n \geq 4$. The inequality is obviously true when $n = 4$, and assuming its validity for given $t \geq 4$, we deduce that

$$2^{(t+1)/2} = 2^{t/2}2^{1/2} \geq 2^{1/2}t = t + t(2^{1/2} - 1) = t + \frac{t}{2^{1/2}+1} \geq t + \frac{t}{3} > t + 1,$$

implying its validity for $t + 1$.

Hence $m = 2^s q \geq 2^s \geq n$, and

$$n! + 1 = (n+1)^m \geq (n+1)^n,$$

which is absurd, since $n! = n(n-1)(n-2)\ \ ...\ \ 2.1$ is clearly less than $n^{n-1}$, for $n > 2$.

This contradiction shows that $k \leq 2$.

The statement holds for $k = 1$ (there $n + 1 = 3 = 2! + 1$). and for $k = 2$ (there $n = 4, n + 1 = 5; 4! + 1 = 5^2$.

So the answer to the question is: $k = 1$ and $k = 2$.