UCD School of Mathematics and Statistics

MATHS SPARKS

PROBLEM SOLVING WORKSHOPS

VOLUME 3

Science Foundation Ireland
For what's next

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Contents

Welcome to the third volume of the UCD Maths Sparks Problem Solving Workshops booklet.

Maths Sparks is a series of workshops for senior-cycle post-primary students, designed and run by volunteer undergraduate students and academic staff of the UCD School of Mathematics & Statistics. The workshops were created to encourage more young people to consider pursuing mathematics and mathematics-based courses at third level, through demonstration of the applications of mathematics outside of the Irish post-primary school curriculum. Through interactive workshops, based on sense-making and students' articulation of their mathematical thinking, these workshops can provide students with an opportunity to explore topics within the broad subject of mathematics and potentially impact their attitudes towards and self-confidence in mathematics.

We hope this booklet will be used by teachers to encourage their students to investigate and explore mathematical topics. As additional activities, teachers may also like to encourage their students to research the various mathematicians noted throughout the booklet. Realising that mathematics is a human endeavour, and not a static body of knowledge, may 'spark' the interest of students and encourage them to pursue their study of mathematics after their post-primary education.

Further information on Maths Sparks and the first two volumes of the Maths Sparks workshops are available at: https://www.ucd.ie/mathstat/mathsparks/

If you would like to share any feedback or commentary on the workshops, please feel free to contact the programme directors (emails below).

Maths Sparks was supported by SFI Discover, UCD College of Science, UCD Access & Lifelong Learning and UCD School of Mathematics & Statistics.

References:

Cronin, A., Ni Shuilleabhain, A., Lewanowski-Breen, E., & Kennedy, C. (2017). Maths Sparks: Investigating the impact of outreach on pupil's attitudes towards mathematics. MSOR Connections, 15(3), 4-13.

Ni Shuilleabhain, A., & Cronin, A. (2015). Maths Sparks: Developing Community and Widening Participation. MSOR Connections, 14(1), 43-53.

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The Josephus Problem

Introduction

The Josephus problem is based around Josephus Flavius; a Jewish soldier and historian who inspired an interesting set of mathematical problems. In 67 C.E., Josephus and 40 fellow soldiers were surrounded by a group of Roman soldiers who were intent on capturing them. Fearing capture, they decided that they would kill themselves instead and whilst Josephus did not agree with this proposal, he was afraid to disagree. Instead, he suggested that they arrange themselves in a circle and, counting around the circle in a clockwise direction, every second man should be killed until there was only one survivor, who would then kill himself. Josephus wanted to be that man (so he could survive!) and he, therefore, had to figure out which position to stand in. Identifying this position is the problem we consider today.

Aim of Workshop

The aim of this workshop is to introduce the students to combinatorics and pattern recognition via the Josephus problem. Students will also be provided with the opportunity to simulate the Josephus problem in the hope of discovering the general formula themselves.

Learning Outcomes

By the end of this workshop, students will be able to:

- Recognise numerical patterns
- Solve problems using powers of two
- Derive the general formula for the Josephus problem
- Calculate the winning position for the Josephus problem

Materials and Resources

One activity sheet per student (with one third of the class on Activity 1A, one third on Activity 1B and one third on Activity 1C).

KEY WORDS

Combinatorics

A branch of mathematics that studies combinations of objects taken from a finite set

The Josephus Problem: Workshop Outline

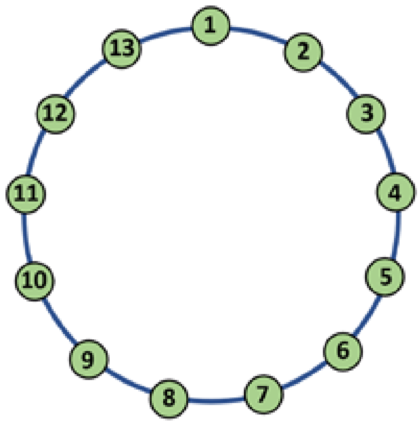
Suggested Time (Total mins)	Activity	Description
5 mins (00:05)	Introduction to the Josephus Problem	· Introduce the Josephus problem (see **Workshop Introduction**) · Demonstrate the idea with 3 volunteers
5 mins (00:10)	Class Activity The Josephus Problem	· Divide students into groups and ask them to apply the Josephus problem with their group - don't try to predict the winning position just yet (see **Appendix – Note 1** for example) · You may wish to go through a full example with the students
15 mins (00:25)	Activity 1 The Josephus Problem	· Hand out one activity sheet per student with one third of the class on Activity 1A, one third on Activity 1B and one third on Activity 1C · **Activity Sheet 1**: Students are asked to find out what position they should be in to survive in each of the given circles (see **Appendix - Note 2**)
15 mins (00:40)	Deduce the formula	· Collate the information into a table on the whiteboard. Have students spotted a pattern? · Discuss the ideas with the class · Deduce the formula using student input (see **Appendix – Note 3**)
10 mins (00:50)	Proof by induction (Optional)	· Explain how the formula could be proved by induction (see **Appendix – Note 4**)
5 mins (00:55)	Conclusion	· Briefly recap the problem, showing how the formula can be used to speed up the process · Encourage students to now find the solution to the original Josephus problem with 41 soldiers
5 mins (01:00)	Class Activity (Optional)	· Everybody in the room stands in a circle. Who survives now?

The Josephus Problem: Workshop Appendix

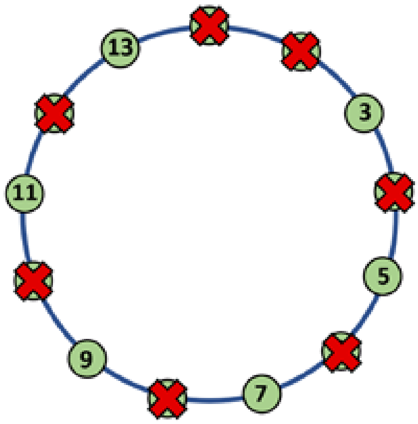
Note 1: Example of the Josephus Problem for 13 soldiers

(A round ends after the person in position n is killed or kills the person beside them)

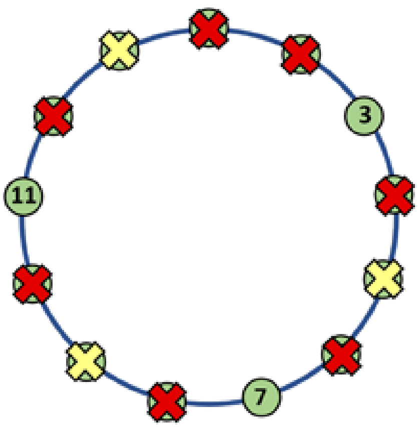
Josephus Problem Example (n=13)



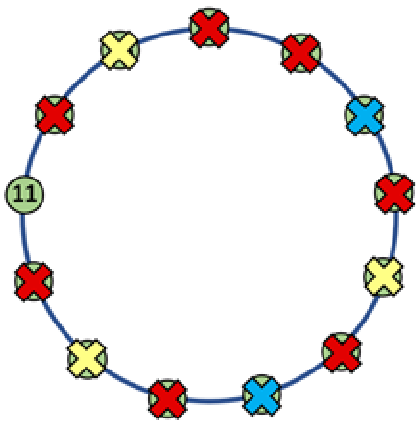
Since every second person is killed in this version of the Josephus problem, we want to find the position of survival. In this example, there are 13 people in the circle.



First round: Starting at 1, moving in a clockwise direction, we kill every second person by crossing them out as shown in red. 1 kills 2, 3 kills 4, 5 kills 6,…,13 kills 1.



Second round: The second round of killings is shown in yellow. This time 3 kills 5, 7 kills 9, and 11 kills 13.



Third round: The final round of killings is shown in blue. 3 kills 7, and 11 kills 3. 11 is, therefore, the position of survival (i.e. the last green circle).

Note 2: Solutions for Activity 1

There are 3 different versions of Activity Sheet 1, each containing circles of varying numbers of soldiers. Each student should work through one of these activity sheets, with the class divided into three groups so that all worksheets are completed.

Each activity sheet has circles of 4, 8, 16 and 32 soldiers to help students identify that a circle with a power of 2 has a '*winning (or survivor) position*' of 1. The rest of the activity sheets are made up of circles containing between 5 and 25 soldiers. When the answers from the three worksheets are collated in a table, as shown below, a pattern begins to emerge (see **Note 3** for formula).

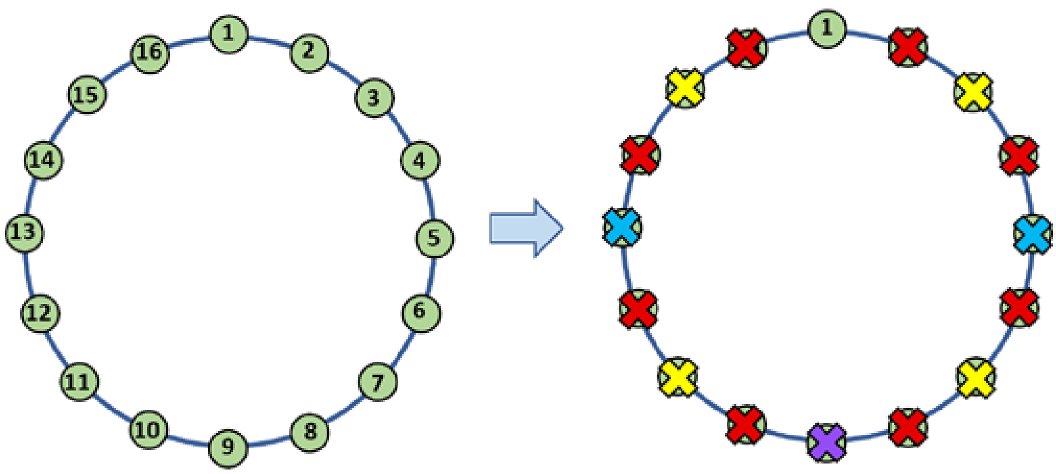
Number of Soldiers	Who Stays Alive?	Number of Soldiers	Who Stays Alive?
4	**1**	**16**	**1**
5	3	17	3
6	5	18	5
7	7	19	7
8	**1**	20	9
9	3	21	11
10	5	22	13
11	7	23	15
12	9	24	17
13	11	25	19
14	13	...	...
15	15	**32**	**1**

Note 3: Deducing the Formula

The best way to find the general solution to the Josephus problem is to map out some scenarios. In this workshop, we have started with 1, 2, 3, 4, 5, …, 10 and once we map the winning position in each case, a pattern begins to emerge.

Number of Soldiers	Winning Position
1	**1**
2	**1**
3	3
4	**1**
5	3
6	5
7	7
8	**1**
9	3
10	5

As indicated in the table above, none of the *winning positions* are even numbers. This is due to the fact that all the people in even-numbered positions are killed in the first round. We also notice that the winning position resets (that is, goes back to 1) for 1, 2, 4, and 8 soldiers. Whilst it intuitively makes sense for 1 to be the *winning position* when there are only 1 or 2 people in the circle, it may not be immediately clear for 4 or 8. However, 4 and 8 are both powers of 2 so we can check whether it resets at each *power* of 2. When we perform the Josephus elimination for 16 (2^4) soldiers, for example, 1 is again the *winning position*. The same is true for 32 (2^5) soldiers and this holds for 2^n soldiers.



To find the general formula, we can use the fact that any number can be written as a power of 2 plus a remainder (r).

Note: after r steps, whoever's turn it is will be the winner as we will be left with a power of 2.

When there are 5 soldiers, the winning position is 3.

If we subtract the next largest $2^n \leq 5$ from the number, we get:

$5 - 2^2 = 1$ remainder (r)

$2(1) = 2$ (Since we kill every second person, we multiply r by 2 to find the position in the circle after r killings)

$2 + 1 = 3$ (we add 1 as we are beginning from the 1^{st} position rather than the 0^{th} position)

3 is the winning position

When there are 20 soldiers, the winning position is 9.

$20 - 2^4 = 4$ remainder

$2(4) = 8$

$8 + 1 = 9$

9 is the winning position

The general formula is thus:

$N - 2^n = r$ (remainder)

$2(r) = 2r$

$2r + 1$ is the winning position

Note 4: Proof by Complete Induction

The formula for the Josephus problem can be proved using proof by complete induction. Whilst this is outside the bounds of this workshop, you may wish to discuss it with your students.

We want to prove that if you have N people sitting in a circle and they take turns eliminating the next person in the circle until only one person remains, the person that remains will be in position 2r + 1, where r is the remainder upon division of N by the largest power of 2 less than or equal to N. We proceed by complete induction on N.

1. First of all, we must prove it for $\mathrm{N} = 1$. Clearly, if there is only one person, they will be the only survivor. Here, $2^0=1$ is the largest power of 2 smaller than or equal to N and thus $\mathrm{r} = 0$ and so $2(0) + 1 = 1$
2. We assume that this result is true for $\mathrm{N} = 1, 2, 3,..., \mathrm{k} - 1$. This is our inductive hypothesis (this is what we mean by Complete Induction).
3. We now need to show that this statement is true for k. To do this, we must consider two possibilities: first of all, when k is even, and secondly when k is odd.

Case 1:

If k is even, it can be expressed as $2^{\mathrm{m}} + \mathrm{r}$ where r is even and $\mathrm{r} < 2^{\mathrm{m}}$ (here we split k into the largest power of 2 less than k and the remainder part which must be even since k and 2^{m} are even).

After the first round of elimination, we will have that all even numbered people are eliminated, and we will be back at person number 1. We can consider this to be a new circle with $2^{\mathrm{m}-1} + \mathrm{r}/2$ people, still starting at 1 and having the same winning person as a circle that size, but with only odd numbered labels for the people in the circle.

By our inductive hypothesis, the survivor is in position $2\,(\mathrm{r}/2) + 1 = \mathrm{r} + 1$ in this new circle. This corresponds to position $\mathrm{r} + (\mathrm{r} + 1) = 2\mathrm{r} + 1$ in the original circle as we had eliminated r people in positions before the $(\mathrm{r} + 1)$ person in the new circle in the first round.

This proves the result for even k.

Case 2:

If k is odd, it can be expressed as $2^m + r$ where r is odd and $r < 2^m$ (again, splitting k into the largest power of 2 less than k plus the remainder part which must be odd since k is odd and 2^m is even).

This time, after the first round of elimination we will have that all even numbered people are eliminated, and the final, odd-numbered person has eliminated number 1. We can consider this to be a new circle only with having $2^{m-1} + (r - 1)/2$ people, starting at 3 (since 1 and 2 are eliminated) and only having odd numbered labels for people in the circle.

Again, by our inductive hypothesis, the survivor is in position $2((r - 1) / 2)$ in this new circle. This corresponds to the position $(r + 1) + r = 2r + 1$ in the original circle as we have eliminated r people in even positions and 1 person in an odd position (namely person number 1), in the positions before the r^{th} person in the new circle in the first round. This proves the result for odd k.

We can now apply complete induction to conclude that the winning position for any $N = 2^m + r$, where $r < 2^m$, is position $2r + 1$.

Sources and Additional Resources

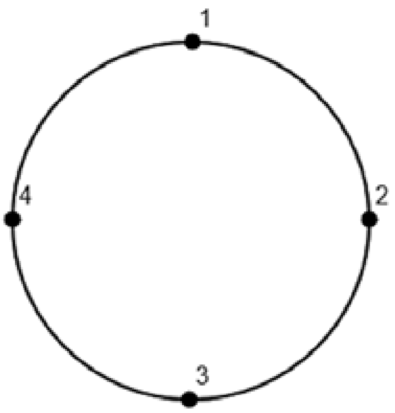
https://www.exploringbinary.com/powers-of-two-in-the-josephus-problem/ (Josephus problem)

https://www.youtube.com/watch?v=uCsD3ZGzMgE (Josephus problem general formula)

The Josephus Problem: Activity 1A

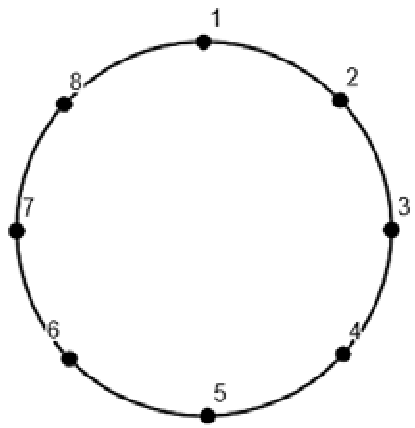
Q1. Find out who stays alive in each of the following scenarios by crossing off every second person in the circle. You may wish to use a different colour for each round (Note: A round ends after the person in position n is killed or kills the person beside them).

A



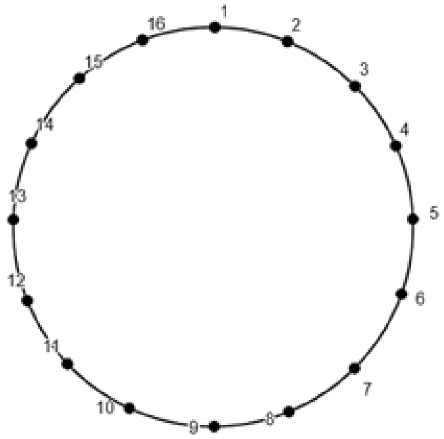
Who stays alive?

B



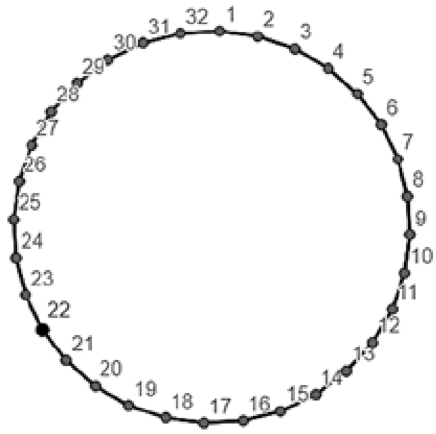
Who stays alive?

C



Who stays alive?

D

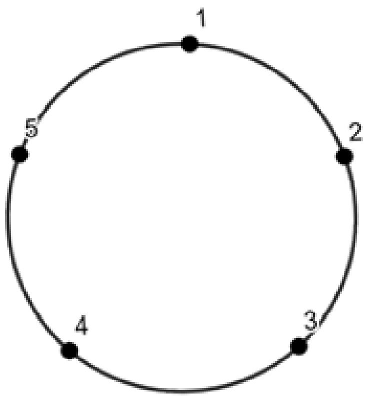


Who stays alive?

Q2. On each diagram A to D rewrite the number of people in the circle in the form 2^n. (Hint: $4 = 2^2$). Can you see a pattern?

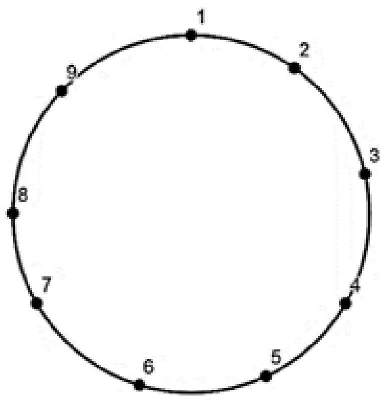
Q3. Please repeat Q1 using the following circles.

E



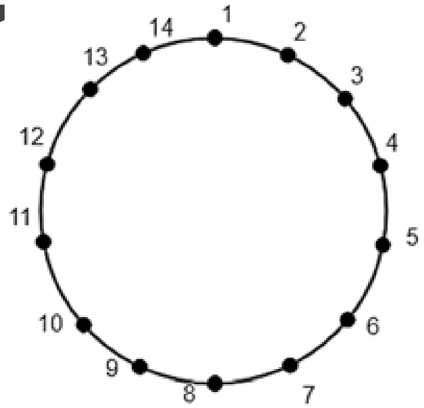
Who stays alive?

F



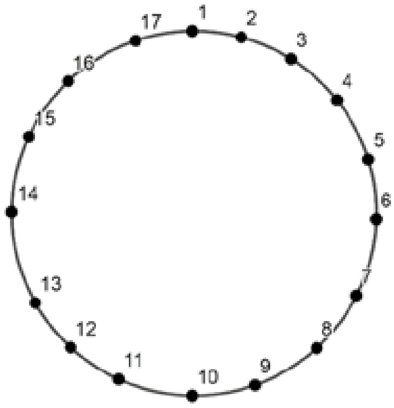
Who stays alive?

G



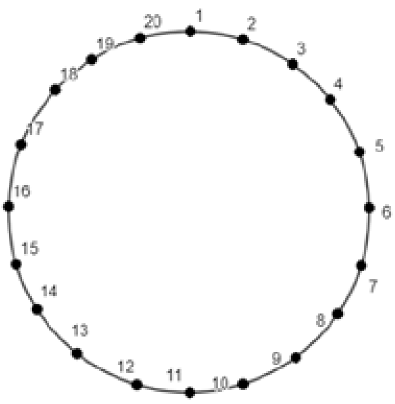
Who stays alive?

H



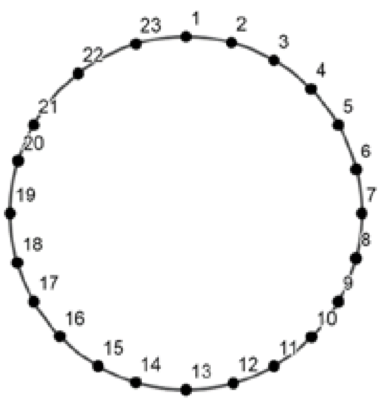
Who stays alive?

I



Who stays alive?

J

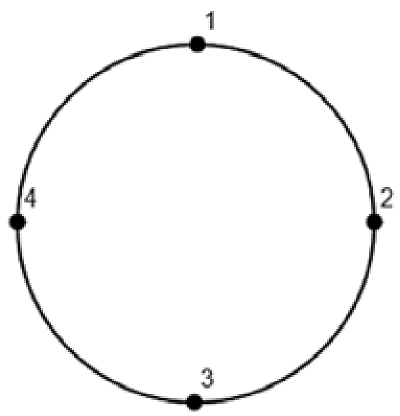


Who stays alive?

The Josephus Problem: Activity 1B

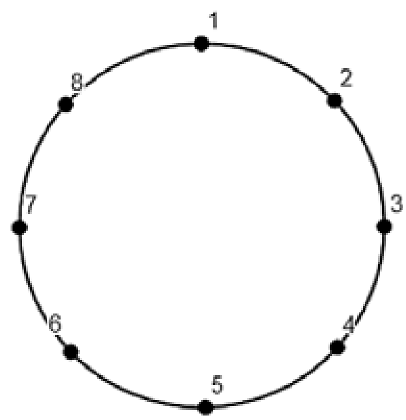
Q1. Find out who stays alive in each of the following scenarios by crossing off every second person in the circle. You may wish to use a different colour for each round (Note: A round ends after the person in position n is killed or kills the person beside them).

A



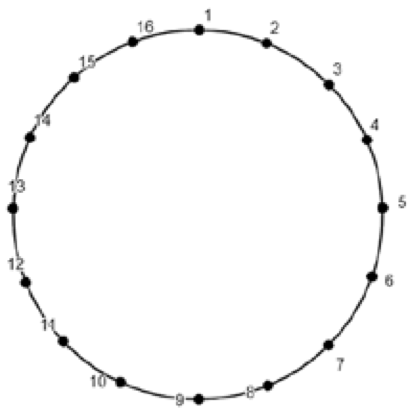
Who stays alive?

B



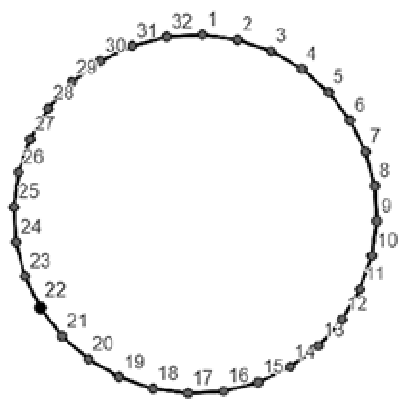
Who stays alive?

C



Who stays alive?

D

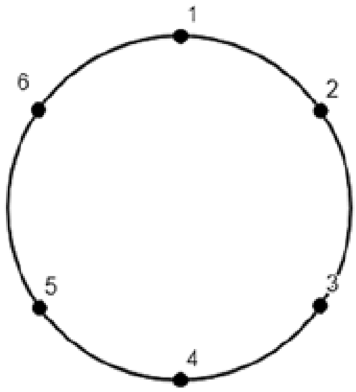


Who stays alive?

Q2. On each diagram A to D rewrite the number of people in the circle in the form 2^n. (Hint: $4 = 2^2$). Can you see a pattern?

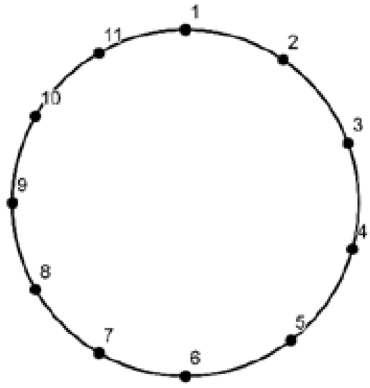
Q3. Please repeat Q1 using the following circles.

E



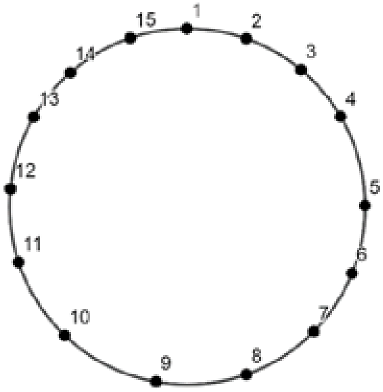
Who stays alive?

F



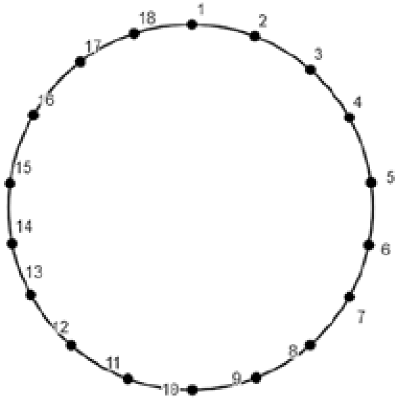
Who stays alive?

G



Who stays alive?

H



Who stays alive?

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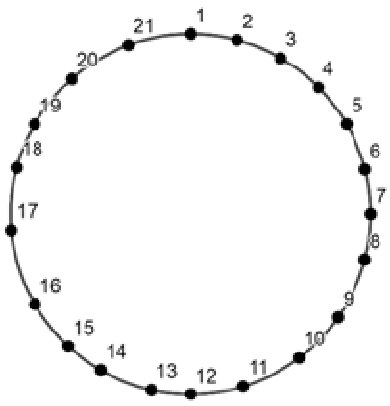
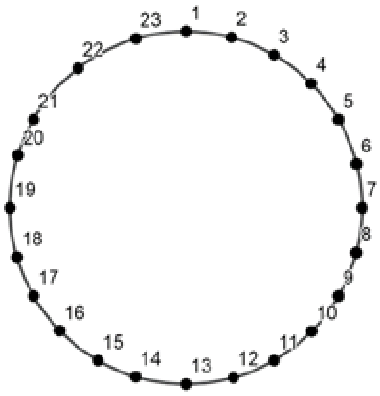


Who stays alive?

J

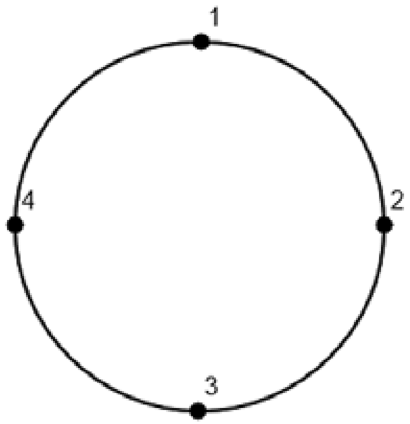


Who stays alive?

The Josephus Problem: Activity 1C

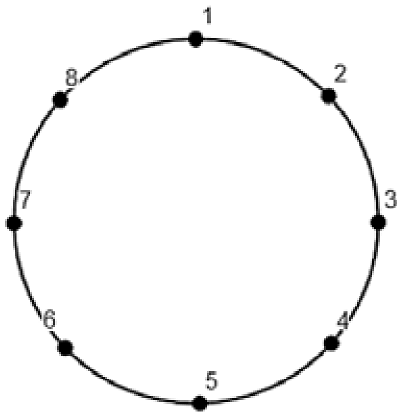
Q1. Find out who stays alive in each of the following scenarios by crossing off every second person in the circle. You may wish to use a different colour for each round (Note: A round ends after the person in position n is killed or kills the person beside them).

A



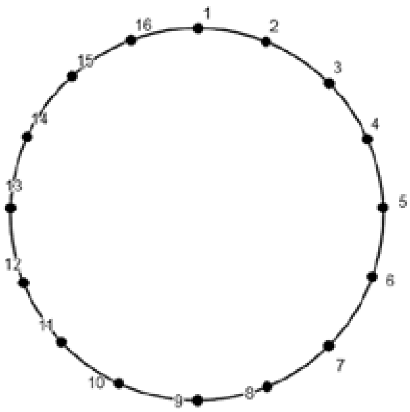
Who stays alive?

B



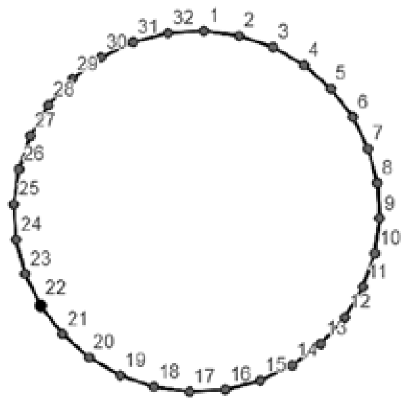
Who stays alive?

C



Who stays alive?

D

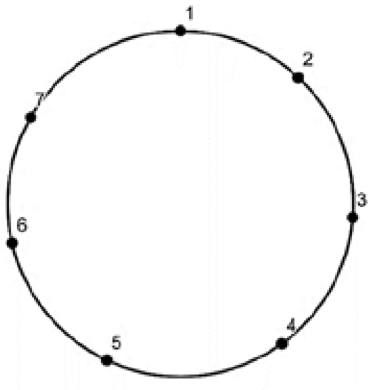


Who stays alive?

Q2. On each diagram A to D rewrite the number of people in the circle in the form 2^n. (Hint: $4 = 2^2$). Can you see a pattern?

Q3. Please repeat Q1 using the following circles.

E



Who stays alive?

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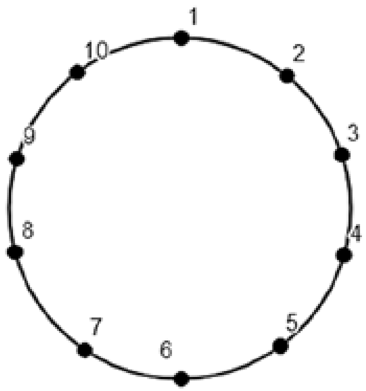

Who stays alive?

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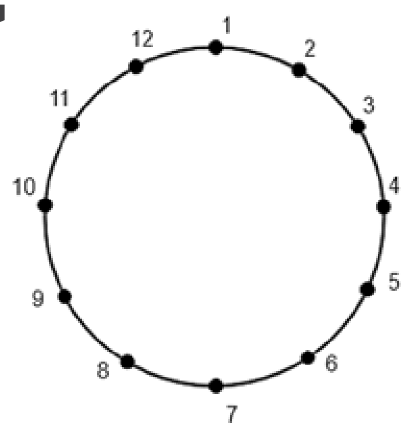

Who stays alive?

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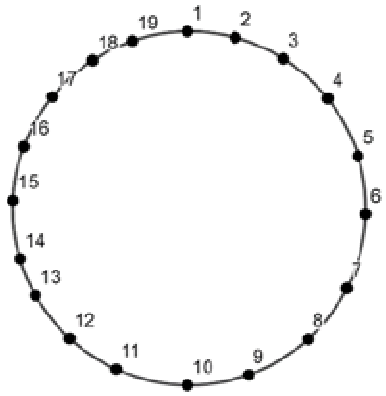

Who stays alive?

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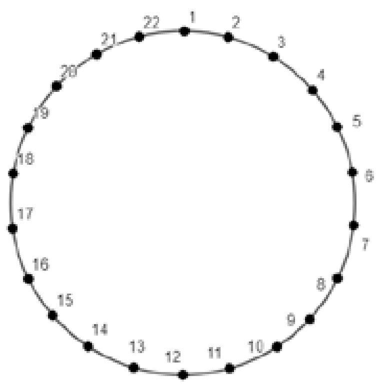

Who stays alive?

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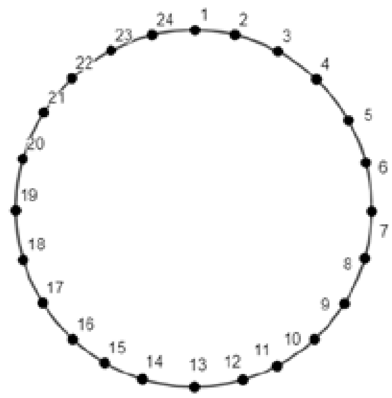


Who stays alive?

Origami

Introduction

Origami is the ancient art of paper folding and whilst this technique is often associated with Japanese culture, its exact origins are unclear. In fact, paper folding has a history that spans a wide range of different cultures from across the globe. However, the true beauty of origami lies not only in the fact that it can be used to create magical structures and paper animals, but it also has applications in mathematics. The rules of paper folding, for example, are encoded in several mathematical axioms known as the Huzita–Hatori axioms. Several geometric construction problems, such as trisecting an angle or doubling a cube, can also be solved using only a few paper folds.

The mathematics of origami has also been extended to real-world applications. The Miura fold, for example, is a highly effective way of folding a piece of paper into a much smaller area and is implemented in space missions to deploy solar panels. While a high school student, Britney Gallivan derived a formula for the maximum number of times a piece of paper can be folded in half and proved the answer to be 12.

Aim of Workshop

The aim of this workshop is to demonstrate mathematical proofs using origami, which the students can physically construct and observe. In particular, students will have the opportunity to prove Pythagoras' theorem using a single sheet of origami paper. The limitations of traditional constructions using a straightedge and compass will also be explored.

Learning Outcomes

By the end of this workshop, students will be able to:

- Construct squares and triangles to prove Pythagoras' theorem
- Justify why $\sqrt[3]{2}$ is not constructible using a straightedge and compass
- Construct $\sqrt[3]{2}$ using origami
- Recognise that several proofs to one problem may exist

Materials and Resources

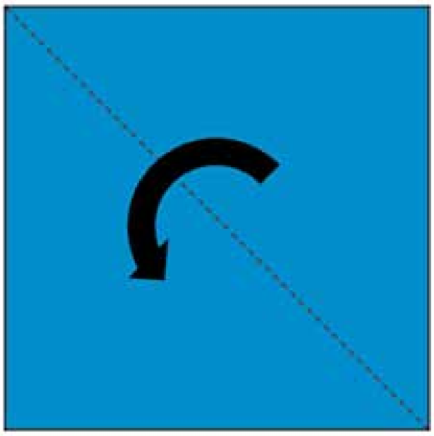
Origami paper, activity sheets

Origami: Workshop Outline

Suggested Time (Total mins)	Activity	Description
5 mins (00:05)	Introduction to the art of paper folding	· Introduce the mathematics of origami (see **Workshop Introduction)**. · Mention that the aim of the workshop is to prove a prominent theory in mathematics using origami
20 mins (00:25)	Activity 1 Proof by origami	· Demonstrate the folding technique by following the steps alongside the students (see **Appendix – Note 1**) · **Activity Sheet 1:** Using their folded origami paper, students attempt to prove the theorem of Pythagoras (see **Appendix – Note 2**)
5 mins (00:30)	Doubling a cube	· Introduce the 'doubling a cube' problem, also known as the Delian problem (see **Appendix – Note 3**)
15 mins (00:45)	Activity 2 Calculating the volumes of cubes	· **Activity Sheet 2:** Students calculate the volumes of various cubes (see **Appendix – Note 4**) · Note: The final questions involve cubed roots and may need to be revised following a discussion of roots
10 mins (00:55)	$\sqrt{2}$ and $\sqrt[3]{2}$	· Ask the students to discuss whether $\sqrt{2}$ is constructible with a straightedge and compass (It is – we can use Pythagoras' theorem!) · Explain that $\sqrt[3]{2}$ cannot be constructed with a straightedge and compass – link back to Activity Sheet 2 (see **Appendix – Note 3**)
5 mins (01:00)	Activity 3 Constructing $\sqrt[3]{2}$ with origami	· Use origami to demonstrate the "construction" of $\sqrt[3]{2}$ by a marked straightedge · Encourage students to measure lengths a and b, and to calculate the ratio.

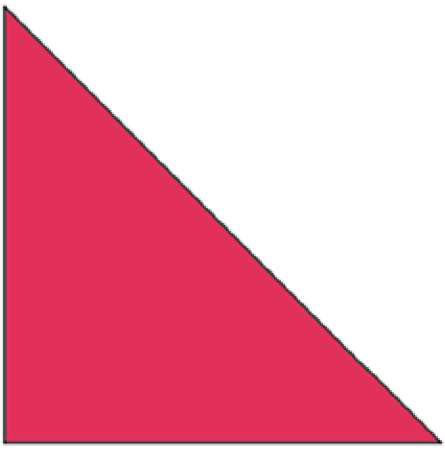
Note 1: Origami Paper Folding Instructions

1



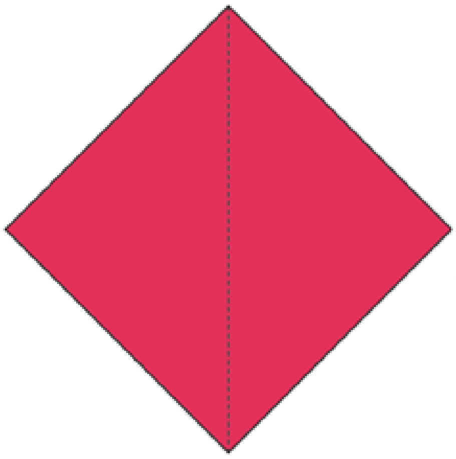
Fold your paper in half along the dotted line as shown.

2



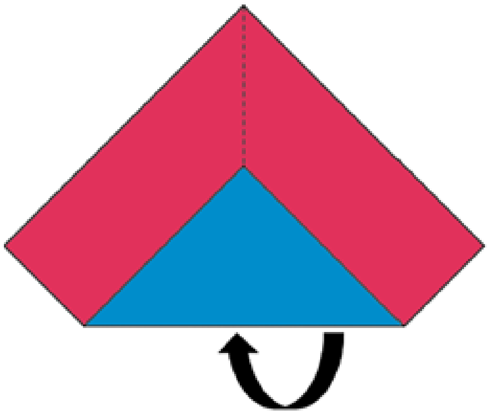
This will be the result.

3



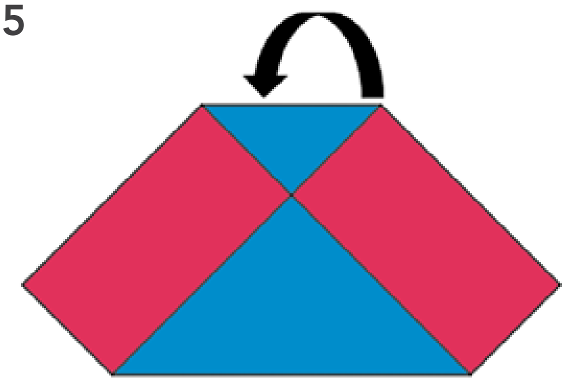
Open the paper and position it like this.

4



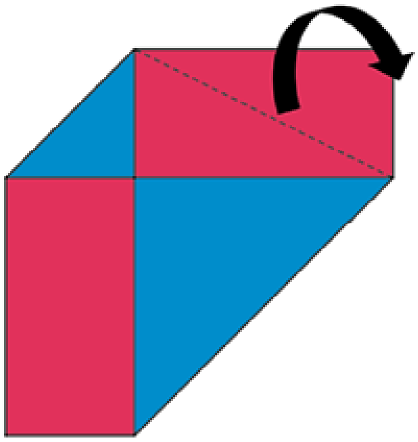
Fold up from the bottom. The triangle you make can be any size that is less than half your original paper.

5



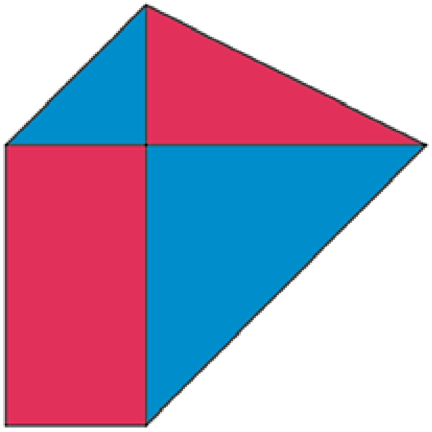
Fold down from the top. The new triangle needs to meet the triangle from the previous step.

6



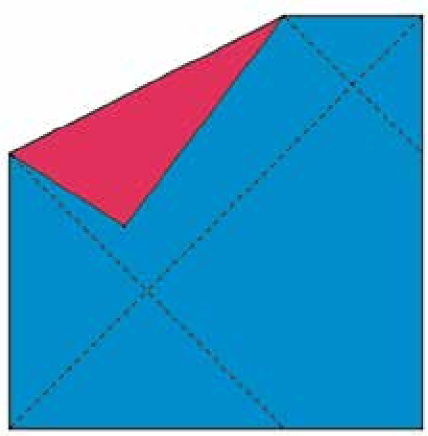
Rotate your paper. Fold along the dotted line to put a triangle behind your paper.

7



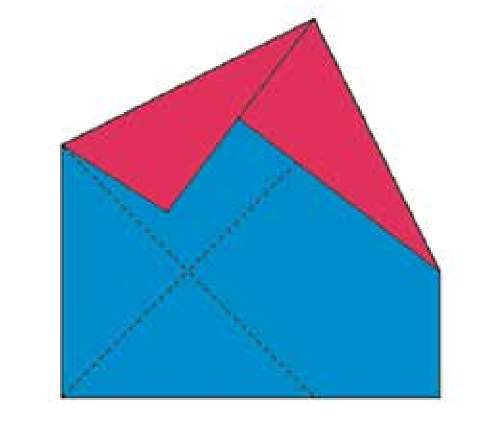
This is what you should have now after folding back the triangle.

8



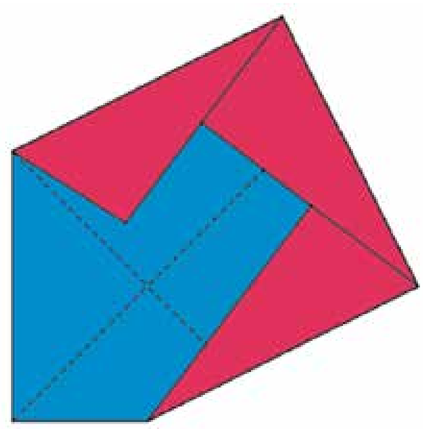
Flip over your paper. Unfold everything except the triangle you just made.

9



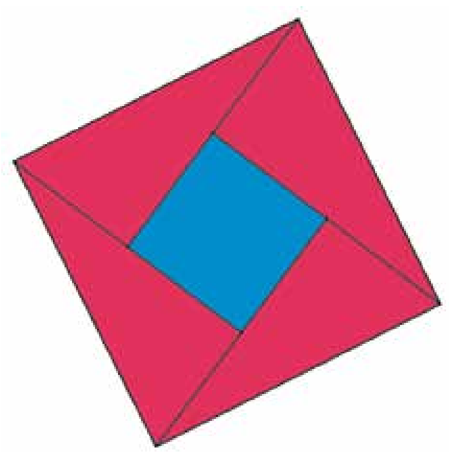
Fold down the top right hand corner to make a triangle. Ensure that the side of the new triangle matches up with the side of the other triangle.

10



Repeat step 9 with the bottom right corner to make another triangle.

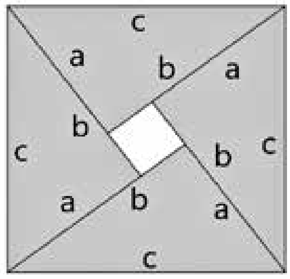
11



Fold in the last corner to make a fourth triangle. Again, make sure that the sides of the triangles meet each other. This is your final product!

Note 2: Solutions for Activity 1

Q1. Fill in the blanks:



(i) A right-angled triangle is a triangle that has one angle measuring 90 degrees.

(ii) The side of the triangle opposite the right angle is the longest side and is called the hypotenuse.

Q2. What is the area of the bigger square (the entire square piece of paper you ended up with after folding)?

The area of the bigger square is c^2 since each side of the square is length c

Q3. What is the area of the smaller square?

$(b - a)^2$

Q4.

(i) What is the area of one of the four identical triangles?

1/2 (ab) (Recall; the area of a triangle can be calculated by multiplying the base by the perpendicular height and dividing by two)

(ii) What is the area of the four triangles together?

2ab

Q5. In words, what is the relationship between the area of the bigger square, the area of the smaller square and the area of the four triangles?

Area of bigger square = Area of smaller square + Area of four triangles

Q6. Use this relationship and your other answers to derive an important theorem in mathematics!

$c^2 = (b - a)^2 + 2ab = b^2 - 2ab + a^2 + 2ab = a^2 + b^2$

(Pythagoras' theorem)

Note 3: Doubling a Cube

'Doubling a cube' (or the Delian Problem) is an ancient geometric problem which involves the construction of the edge of a cube whose volume is double that of a given cube.

This problem owes its name to a Greek legend concerning the inhabitants of Delos who were suffering from a terrible plague that was ravaging their island. The inhabitants believed that Apollo, the god of healing, had purposely sent the plague to kill them and so they sought guidance from the oracle (priest) at Delos on how to appease the gods. The oracle explained that Apollo was furious because the altar in the temple of Delos was too small and therefore instructed them to double it in size. The people of Delos immediately rushed to the temple to construct a new alter that was twice as wide, twice as long and twice as tall as the previous one in the hope of saving their island. However, Apollo was still not pleased as the new altar was now eight times the size of the original, rather than twice the size. As a consequence, the plague continued to spread throughout the island of Delos, claiming the lives of its inhabitants.

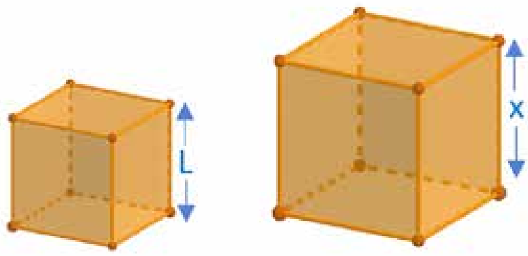
This legend demonstrates that doubling the dimensions of a cube will not double the volume.

In order to double the volume of a cube, with side length L, we would need:

- (Length of larger cube)^3 = $2(L^3)$
- Let the length of the larger cube be x.

We, therefore, have that $x^3 = 2(L^3)$

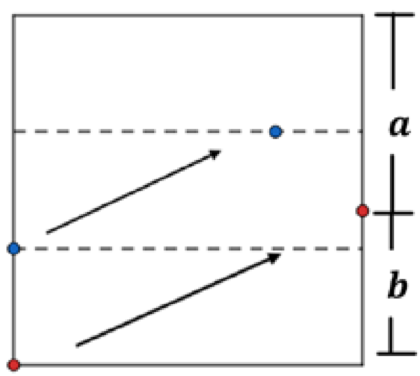
- Solving for x, we see that $x = \sqrt[3]{2}\ L$.



In other words, the length of the new cube would need to be $\sqrt[3]{2}$ times the length of the original cube. However, $\sqrt[3]{2}$ is an irrational number ($\approx 1.259921\ldots$) and we, therefore, cannot accurately construct a cube that is double the volume of another using a straightedge and compass.

However, it is possible to construct this ratio using origami:

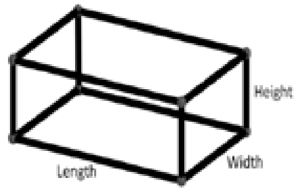
1. Fold a square piece of paper into thirds as shown below.
2. Fold the paper so that the bottom third (blue dot) touches the two-thirds line at the same time as the bottom left-hand corner (red dot) touches the right edge of the paper
3. The ratio between a and b is $\sqrt[3]{2}$. Therefore, a cube with a side length of "a", will have twice the volume of a cube of side length "b". We can see this by rearranging $a/b = \sqrt[3]{2}$ to get $a = \sqrt[3]{2}\,(b)$



$$\frac{a}{b} = \sqrt[3]{2}$$

Note 4: Solutions for Activity 2

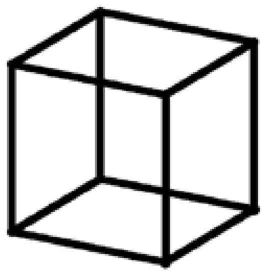
Q1. Label the edges of the cuboid with the following terms: length, width, height



Q2. What is the volume formula for a cuboid?

Volume of cuboid = Length × Width × Height

Q3. Label the edges of the cube



Q4. What is the volume formula for a cube?

Volume of cube $=$ Length $\times$ Length $\times$ Length $=$ Length^3

Q5. What is the volume of the following cube?

$(4\text{ cm})^3 = 64\text{ cm}^3$

Q6. What would the volume of a cube with double (or twice) the volume of the previous cube be?

$(64\text{cm}^3) \times 2 = 128\text{ cm}^3$

Q7. What would the volume of a cube with double the length of the cube in Q5 be?

$(4\text{ cm} \times 2)^3 = (8\text{cm})^3 = 512\text{ cm}^3$

Q8. What do you notice about the answers to Q6 and Q7?

The answers are different; we cannot double the volume of a cube just by doubling its length.

Q9. What would the volume of a cube with double the volume of the following cube be (in terms of L)?

Volume $= 2L^3$

Q10. We have a cube with a side of length y cm. We want our cube to have the same volume as the cube in the answer to Q6. What do we need the value of y to be?

$y = \sqrt[3]{128} \approx 5.04$

Q11. We have a cube with a side of length x cm. Now we want our cube to have the same volume as the cube in the answer to Q9. What do we need the value of x to be?

$x = \sqrt[3]{(2L^3)} = \sqrt[3]{2}\ (L)$

Sources and Additional Resources

https://www.tor.com/2017/06/29/the-magic-and-mathematics-of-paper-folding/ (Mathematics of origami)

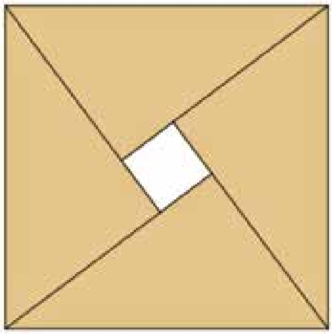
https://youtu.be/R4IMaeZmgLA (Proof of Pythagoras using origami)

https://www.youtube.com/watch?v=4Ncc5A2xT78 (Delian problem)

http://www.cutoutfoldup.com/409-double-a-cube.php (Constructing $\sqrt[3]{2}$ using origami)

Origami: Activity 1

Having completed the folds, you should end up with something like this. Don't worry if your inner square is bigger or smaller than someone else's!



Q1. Fill in the blanks:

(i) A right-angled triangle is a triangle that has one angle measuring _______________degrees.

(ii) The side of the triangle opposite the right angle is the longest side and is called the

____

Instructions:

- Label this longest side in each of the four triangles you have folded **c**. These will also be the four sides of the large square.
- Label the other two sides in the four triangles **a** and **b**. Call the smallest side **a** and the other side **b**.

Q2. What is the area of the bigger square (the entire square piece of paper you ended up with after folding) in terms of the variables above?

Q3. What is the area of the smaller square in terms of the variables?

Q4.

(i) What is the area of one of the four identical triangles in terms of the variables?

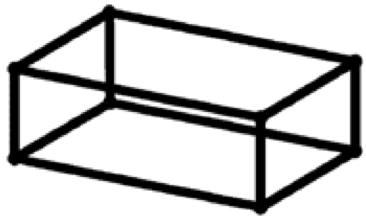
(ii) What is the area of the four triangles together in terms of the variables?

Q5. In words, what is the relationship between the area of the bigger square, the area of the smaller square and the area of the four triangles?

Q6. Use this relationship and your other answers to derive an important theorem in maths!

Origami: Activity 2

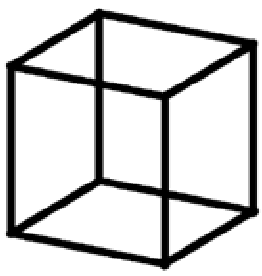
Q1. Label the edges of the cuboid with the following terms: Length, Width, Height



Q2. What is the volume formula for a cuboid?

Volume of cuboid =

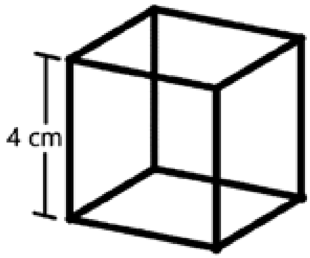
Q3. Label the edges of the cube.



Q4. What is the volume formula for a cube?

Volume of cube =

Q5. What is the volume of the following cube?



Q6. What would the volume of a cube with double (or twice) the volume of the previous cube be?

Q7. What would the volume of a cube with double the length of the cube in Q5 be?

Q8. What do you notice about the answers to Q6 and Q7? Are they the same?

Q9. What would the volume of a cube with double the volume of the following cube be in terms of L?

Q10. We have a cube with a side of length y cm. We want our cube to have the same volume as the cube in the answer to Q6. What do we need the value of y to be?

Q11. We have a cube with a side of length x cm. Now we want our cube to have the same volume as the cube in the answer to Q9. What do we need the value of x to be?

World of Markets

Introduction

Stock markets are often considered a complex and alien concept that conjure up images of trading floors full of adrenalin-fuelled buyers and sellers. In fact, we regularly hear the names of global brands, such as Apple or Amazon, but have little understanding of how these companies are given a monetary value or have claimed their reputations as leaders in their fields. Markets can actually be described using common mathematical principles. This workshop will build on students' understanding of statistics, with the aim of demonstrating to them the random nature of stock markets in today's world. In particular, this workshop will introduce students to the theory of random walks by drawing on the concepts of standard deviation and normal distribution.

Aim of Workshop

The aim of the workshop is to build on students' prior learning of statistics in order to introduce them to the workings behind the global stock markets. A real market environment will also be created using well-known brands to spark their interest in the topic.

Learning Outcomes

By the end of this workshop, students will be able to:

- Explain how a stock market works
- Model a random walk
- Describe, in their own words, what is meant by the normal distribution

Materials and Resources

Coins, activity sheets, open space to demonstrate the random walk

KEY WORDS

Market
A place, either physical or virtual, where buyers and sellers meet to exchange goods and services

Security
Rights to assets, mostly in the form of shares, bonds or stock

Stock
The share in the ownership of a company

Share price
The price of a single share of a company

World of Markets: Workshop Outline

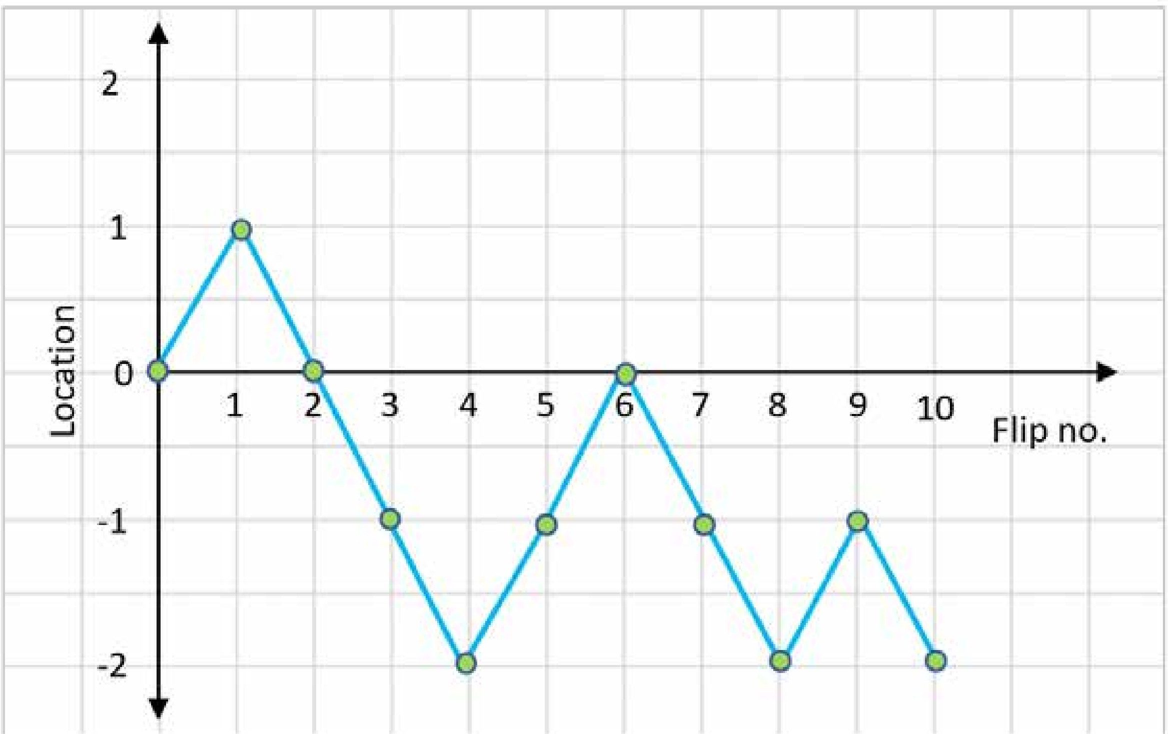
Suggested Time (Total mins)	Activity	Description
5 mins (00:05)	Introduction to stock markets	· Ask students to discuss what comes to mind when they think of a market · Discuss the ideas as a class before introducing the concept of a market (see **Key Words**)
5 mins (00:10)	Securities and Stocks	· Ask students if they can think of any examples of markets (see **Appendix – Note 1**) · Explain what is meant by securities and introduce the idea of a market where securities are traded (see **Key Words**) · Discuss the concept of stock and share price (See **Key Words**).
15 mins (00:25)	Activity 1 Flip Trip	· Hand out the activity sheet and a coin to each pair of students and explain the task · **Activity Sheet 1:** In pairs, students flip a coin 10 times and graph the current total (see **Appendix – Note 2**)
5 mins (00:30)	Random Walk and Stochastic Process	· Explain stochastic processes (see **Appendix – Note 3**) · Give the example of random walk and refer back to Activity 1 (see **Appendix – Note 3**)
10 mins (00:40)	Normal Distribution	· Mention that random walks follow a standard normal distribution and recap the characteristics of a normal distribution · You may wish to revise mean and variance and link this to standard normal distribution
10 mins (00:50)	Class Activity Random Walk	· Optional: Students model a random walk (see **Appendix – Note 4**)
5 mins (00:55)	Conclusion	· Refer to the share price of some well-known brands to show how share prices change (Link this to a random walk)

Note 1: Different Types of Markets

Physical market	Place where goods are physically bought and sold
Virtual market	Place were goods are bought and sold electronically, such as Amazon
Securities market	A financial market where bonds, stocks, shares and other securities are traded

Note 2: Sample Solution for Activity 1

Flip Number		1	2	3	4	5	6	7	8	9	10
H/T	-	H	T	T	T	H	H	T	T	H	T
Value	-	+1	-1	-1	-1	+1	+1	-1	-1	+1	-1
Location	0	+1	0	-1	-2	-1	0	-1	-2	-1	-2



Note 3: Stochastic Processes and Random Walk

A stochastic process is a way of describing successive random events that are associated with a variable such as time. Even if the initial starting point is known, there are often several directions in which the process may evolve. In Activity 1, for example, there was a 50:50 chance of getting heads or tails with each flip of the coin, yet the actual outcomes for each person was not the same. So, whilst we can define certain characteristics for a process, such as probabilities or starting points, we cannot predict what will actually happen and, thus, it is random.

Similarly, the future share prices of a company cannot be determined as it depends on supply and demand. If more people hope to sell a stock than buy it, then there would be a greater supply than demand and thus the price would fall. However, this cannot be predicted and is therefore considered a stochastic process.

A random walk is an example of a stochastic process that consists of a sequence of steps (such as movements in share prices) determined completely by chance. In Activity 1, for example, we started at the origin and moved +1 if the coin landed on heads and -1 if the coin landed on tails. We cannot predict where we will end up after the 10th throw as our movement is determined by chance.

However, we are more likely to end up closer to the origin than at +10 or -10. For larger samples, such as 100 coin tosses, the outcomes will begin to model a standard distribution.

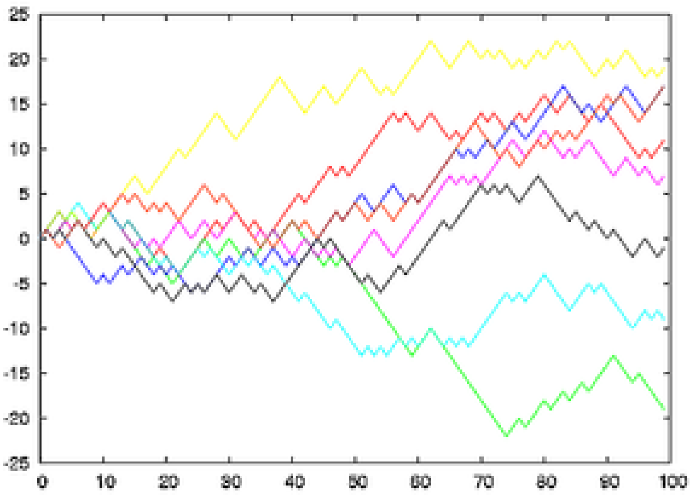


Figure 1 shows a series of random walks, each of which started at the origin.

Note 4: Random Walk Simulation

Step 1: For the random walk simulation, ask students to line up along a central line. This represents the location 0 (Note: you may want to do this outside or in a large room).

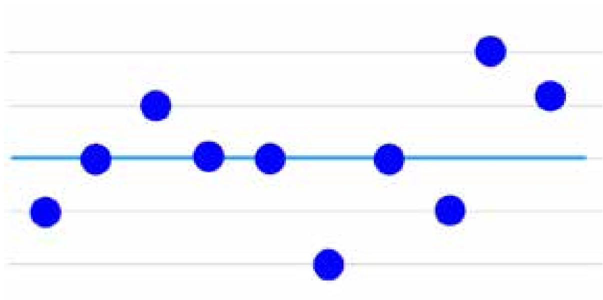
Step 2: Now ask students a series of 5 to 10 binary questions, whereby there are only two possible answers. One answer will correspond to stepping forward one step (+1) and the other will correspond to stepping backward one step (-1). For example, do you have brown hair? Do you prefer summer or winter?

Step 3: Ask students to stay in their final position (e.g. -2) but to move down towards the wall and group together (see diagram below)

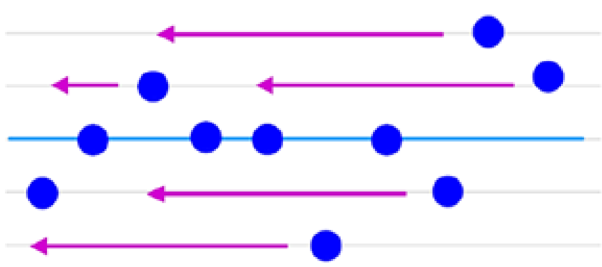
The result of the random walk should follow a normal distribution, with most students centred around 0.

Random Walk Simulation

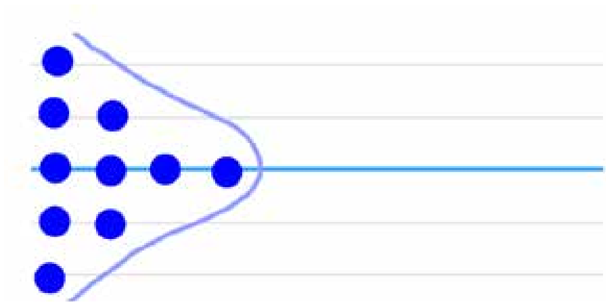
Step 1



Step 2



Step 3



Result

Sources and Additional Resources

https://www.thebalance.com/securities-definition-and-effect-on-the-u-s-economy-3305961 (securities)

https://momath.org/wp-content/uploads/2016/11/Random-Walk-lesson-9.12.15.pdf (random walk)

http://www.randomservices.org/random/ (additional resources)

World of markets: Activity 1

Q1. Flip the coins a total of 10 times and record the events in the table below as follows:

- H/T: Fill in "H" for heads and "T" for tails.
- Value: If you flip heads you will step forward one step (+1) and if you flip tails you will step backward one step (-1).
- Location: You will start at position 0 as seen in the table below. You then add the value from each flip to your previous location to find your new location (e.g. if you land on heads first, followed by another heads, your location is now +2)

Flip Number		1	2	3	4	5	6	7	8	9	10
H/T	-										
Value (+1 or -1)	-										
Location	0										

Q2. Graph your location results on the graph paper. Compare this with others in the class.

Cryptography

Introduction

Note: For a similar workshop, please refer to *'Cryptography'*, Maths Sparks Volume I.

Cryptography is the art of producing or solving codes and has been used as a method of secure communication since as early as 1900 BCE. Whilst Cryptography initially concerned communication and linguistics, it has become an incredibly important area of mathematics given its roots in number theory and its relevance to internet security. One of the most well-known examples of Cryptography in ancient times was the 'Caesar cipher' which was first developed by Julius Caesar and reportedly used to communicate messages across the Roman Empire. The Caesar cipher is considered one of the most simplistic forms of encryption, given that it uses a substitution technique whereby each letter is replaced by another further on in the alphabet. However, frequency analysis can be used to decipher such codes and it is therefore considered a relatively weak and unreliable method of encryption. This being said, the 'Vigenère cipher', which is a variation of Caesar cipher, is a more secure form of communication given that a keyword is used to encrypt the message and thus each letter has a different shift. The 'Pigpen cipher' is a visual cipher, replacing letters with symbols. It was used throughout the American Civil war, as well as by the Freemasons.

Aim of Workshop

This workshop will introduce students to the basic concepts of Cryptography including ciphers, decrypting codes and the use of *modulo arithmetic* in Cryptography. Students will also be provided with the opportunity to create their own encrypted messages, which they can then give to their classmate to solve.

Learning Outcomes

By the end of this workshop, students will be able to:

- Describe historical decryption strategies
- Explain, in their own words, how modular arithmetic works.
- Encrypt and decrypt coded words using the Caesar, Vigenère and Pigpen ciphers

Materials and Resources

Vigenère grid, Pigpen cipher, encryption wheels, activity sheets, computer (optional)

KEYWORDS

Cipher

A way of making a word or message secret by changing or rearranging the letters in the message.

Shift

A value, X, which causes the letters to move X number of spaces up or down the alphabet line.

Cryptography: Workshop Outline

Suggested Time (Total mins)	Activity	Description
10 mins (00:10)	Introduction to Cryptography	· Introduce the concept of Cryptography and outline the history of Cryptography (see **Workshop Introduction**) · Explain what is meant by the term cipher (see **Key Words**)
35 mins (00:45)	Activity 1 The Caesar Cipher	· Introduce modular arithmetic using the example of a clock and the days of the week (see **Appendix – Note 1**) · Explain the Caesar cipher and demonstrate how to encrypt and decrypt words (see **Appendix – Note 2**) · Hand out Activity Sheet 1 and an encryption wheel to each student (**Appendix – Note 4**) · **Activity Sheet 1:** Students encrypt and decrypt various messages using the Caesar cipher (see **Appendix – Note 3**)
25 mins (01:10)	Activity 2 The Vigenère Cipher	· Mention that the Vigenère cipher is a variation of the Caesar cipher and explain how it works using an example on the board (see **Appendix – Note 5**) · Hand out Activity Sheet 2 and the Vigenère table to each student (see **Appendix – Note 8**) · **Activity Sheet 2:** Students encrypt and decrypt various messages using the Vigenère cipher (see **Appendix – Note 6**)
15 mins (01:25)	Activity 3 The Pigpen Cipher	· Explain how the Pigpen cipher works (see **Appendix – Note 7**) · **Activity 3:** Ask students to encrypt messages using the Pigpen cipher and give it to their partner to solve (see **Appendix – Note 9**)
15 mins (01:40)	Kahoot Quiz (Optional)	· **Activity 4:** Students answer questions relating to Cryptography using Kahoot (see **Sources and Additional Resources** for the link and **Appendix – Note 10** for solutions)

Note 1: Modular Arithmetic

Modular arithmetic is a system of counting where we cycle back to the start upon reaching a fixed quantity known as the modulus. Once we reach 12 on a clock, for example, we start back at 1. Therefore, 15:00 on a clock corresponds to 3 modulo 12, denoted 3 mod 12.

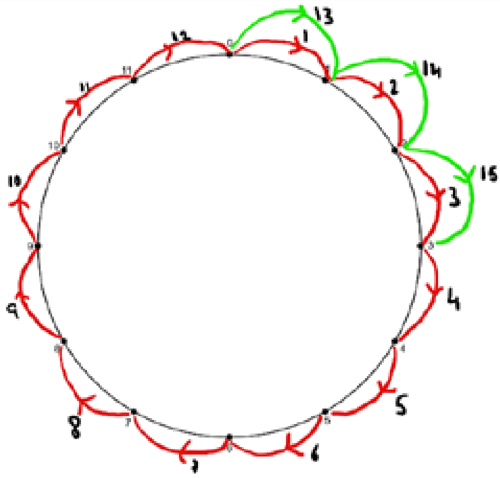
If we are working with mod n, we replace each of the numbers with its remainder when divided by n.

$15 \equiv 12 + 3 \Rightarrow 15 \equiv 3 \text{ mod } 12$

$27 \equiv 2(12) + 3 \Rightarrow 27 \equiv 3 \text{ mod } 12$

$35 \equiv 2(12) + 11 \Rightarrow 35 \equiv 11 \text{ mod } 12$

The same idea applies in Cryptography whereby once the letter Z is reached, we go back to A. This will be demonstrated in the example of Caesar cipher.



Note 2: Caesar Cipher

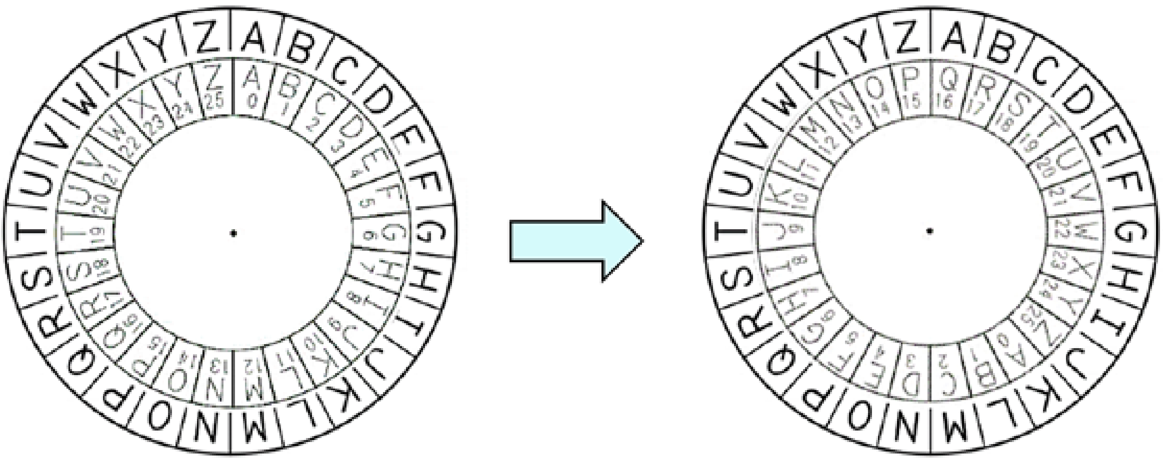
The Caesar cipher was used by Julius Caesar for military messages. This is a very simple cipher where each letter is shifted forward by a common number of places, known as the shift.

In the following example, we want to encrypt the message "Julius Caesar" using a shift of 10:

1. Write down the message to be coded
2. Fill in the number corresponding to the letter (A = 0 and Z = 25)
3. Add the shift to the numbers corresponding to the letters (which is 10 in this example)
4. Reduce your answer mod 26 (since there are 26 letters in the alphabet)
5. Translate these numbers back to letters to find the encrypted message (i.e. 19 = T etc.)

Original	J	U	L	I	U	S	C	A	E	S	A	R
Place no.	9	20	11	8	20	18	2	0	4	18	0	17
Add Shift	19	30	21	18	30	28	12	10	14	28	10	27
Mod 26	19	4	21	18	4	2	12	10	14	2	10	1
Final	T	E	V	S	E	C	M	K	O	C	K	B

To use an encryption wheel, we start off by aligning the inner and outer wheel. We then move the inner wheel n times, where n is the shift. In the example above, the shift is 10 so we rotate the inner wheel 10 places in a clockwise direction. The outer wheel represents the encrypted letter (e.g. the encrypted letter for A is now K, B is now L etc.)
Note: if students are decrypting a coded message, they use the outer wheel and read the corresponding letter on the inner wheel.



Note 3: Solutions for Activity Sheet 1

Q1. Chris wants to encrypt the phrase "ATTACK AT DAWN" using a Caesar cipher and a shift of 10.

	A	T	T	A	C	K	A	T	D	A	W	N
I	0	19	19	0	2	10	0	19	3	0	22	13
II	10	29	29	10	12	20	10	29	13	10	32	23
III	10	3	3	10	12	20	10	3	13	10	6	23
IV	K	D	D	K	M	U	K	D	N	K	G	X

Encrypted message: KDDKMU KD NKGX

Q2. Sally wants to encrypt the phrase "BRUTE FORCE ATTACK" by a shift of 5.

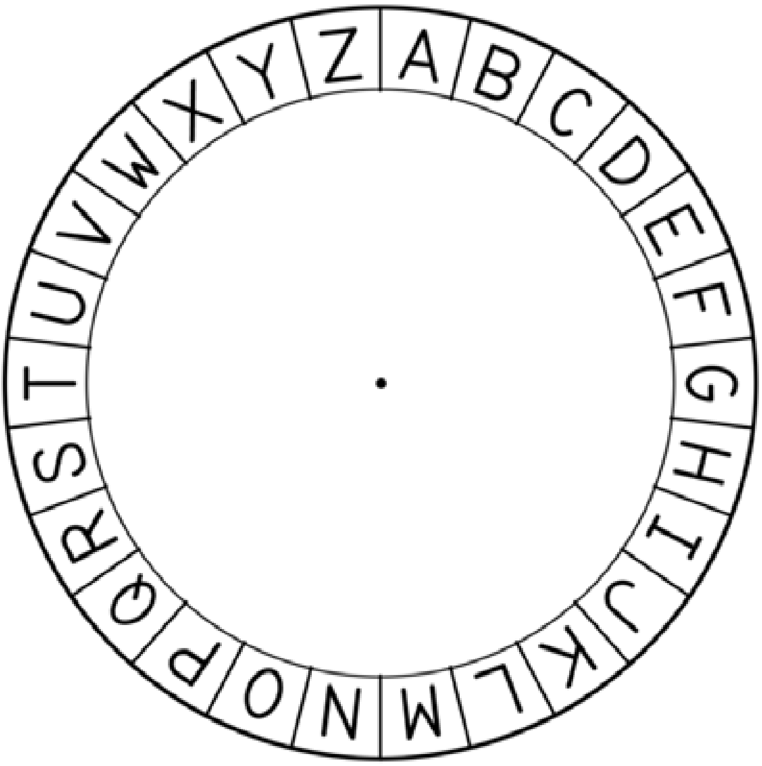
B	R	U	T	E	F	O	R	C	E	A	T	T	A	C	K
G	W	Z	Y	J	K	T	W	H	J	F	Y	Y	F	H	P

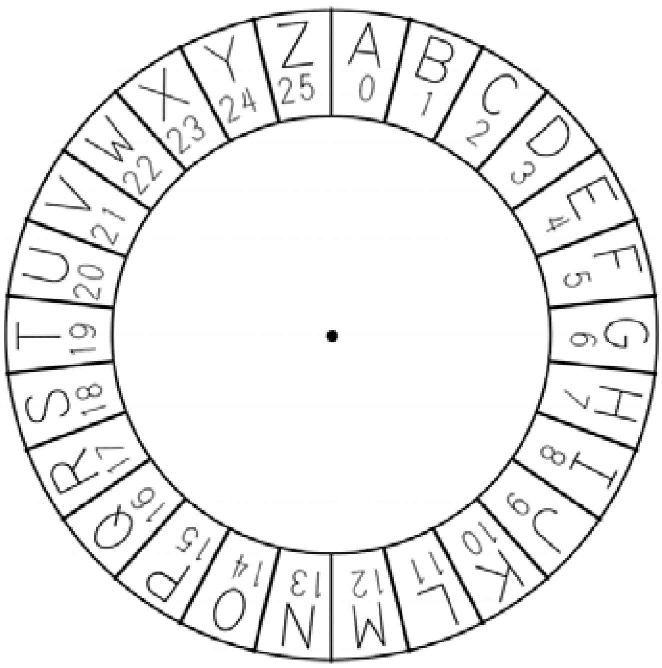
Q3. Mohammed wants to decode "VJCQB RB ODW" using a shift of 9. Using your wheel, can you decrypt the message?

V	J	C	Q	B	R	B	O	D	W
M	A	T	H	S	I	S	F	U	N

Decrypted message: Maths is fun

Note 4: Encryption Wheel Template





Note 5: Vigenère Cipher

The Vigenère cipher, also referred to as 'le chiffre indechiffrable', is a variation of the Caesar cipher which uses a keyword to encrypt the message and thus, each letter has a different shift. In the following example, we want to encrypt the message: "Vigenere" using the keyword "Key":

1. Write down the message to be coded
2. Include the corresponding numbers beside the letters to be coded (A = 0 and Z = 25)
3. Write in the keyword underneath, repeating the keyword if necessary
4. Include the corresponding numbers beside the letters in the keyword (A = 0 and Z = 25)
5. Add the value of the keyword letters to the original letters (each letter will have a different shift)
6. Reduce your answer mod 26
7. Translate these numbers back to letters to find the encrypted message (i.e. 5 = F etc.)

Original	V (21)	I (8)	G (6)	E (4)	N (13)	E (4)	R (17)	E (4)
Keyword	K (10)	E (4)	Y (24)	K (10)	E (4)	Y (24)	K (10)	E (4)
Add Shift	31	12	30	14	17	28	27	8
Mod 26	5	12	4	14	17	2	1	8
Final	F	M	E	O	R	C	B	I

Original	V	I	G	E	N	E	R	E
Keyword	K	E	Y	K	E	Y	K	E
Final	F	M	E	O	R	C	B	I

Alternatively, you can use the Vigenère square to encrypt messages using a keyword:

1. Write down the original message
2. Fill in the keyword underneath
3. Using the Vigenère table, find the first letter of the keyword along the top row (in this case K)
4. Find the letter in this column that is also in the row associated with the corresponding letter of the original phrase (in this case V). This gives us our first encoded letter (i.e. F)
5. Continue for each of the letters in the message.

Keyword

Original

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
A	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
B	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A
C	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B
D	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C
E	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D
F	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E
G	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F
H	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G
I	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H
J	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I
K	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J
L	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K
M	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L
N	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M
O	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N
P	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
Q	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
R	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q
S	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
T	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
U	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T
V	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U
W	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	D
X	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	C
Y	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	B
Z	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y

Note: To decrypt a message, we find the letter of the keyword in the first row and identify the encrypted letter in the same column (in this case F). Now read across to find the original letter.

Note 6: Solutions for Activity 2

Q1. Using the keyword "CIPHER", encrypt the phrase "OVER AND OUT".

	O	V	E	R	A	N	D	O	U	T
I	C	I	P	H	E	R	C	I	P	H
II	Q	D	T	Y	E	E	F	W	J	A

Encrypted message: GWZYJ KTWHJ FYYFHP

Q2. Decrypt the phrase "FIEQ BYOY FROL" using the keyword "MONDAY".

	F	I	E	Q	B	Y	O	Y	F	R	O	L
I	M	O	N	D	A	Y	M	O	N	D	A	Y
II	T	U	R	N	B	A	C	K	S	O	O	N

Decrypted message is: Turn back soon

Q3. Decrypt the phrase "UT BZ JMIGPFS" using the keyword "MATHS".

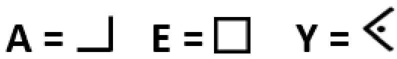
	U	T	B	Z	J	M	I	G	P	F	S
I	M	A	T	H	S	M	A	T	H	S	M
II	I	T	I	S	R	A	I	N	I	N	G

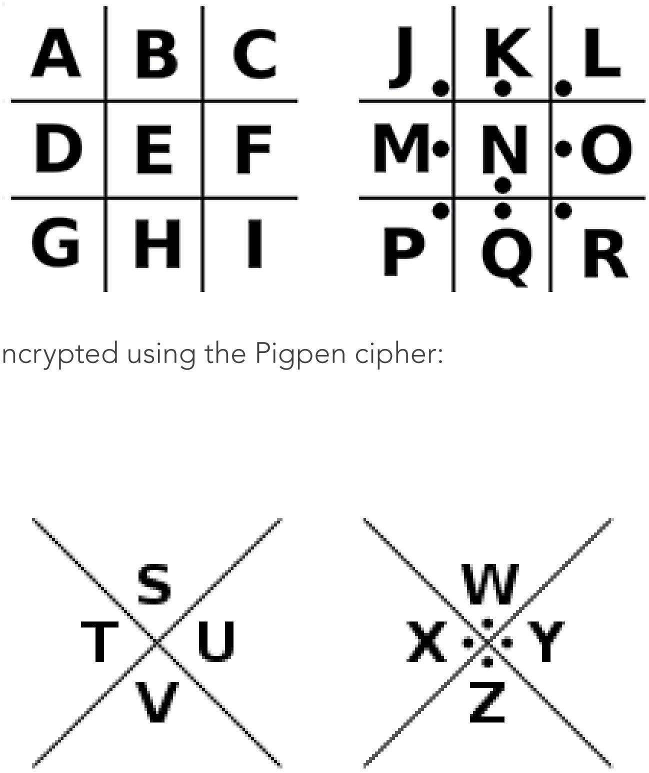
Decrypted message is: It is raining

Note 7: Pigpen Cipher

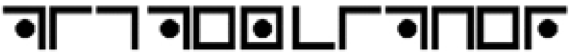
In order to encrypt a message using the Pigpen cipher, the alphabet is written in grids as shown below and each letter is then encrypted by replacing it with a symbol that corresponds to the portion of the grid containing the letter (i.e. the borders for each letter).

For example:





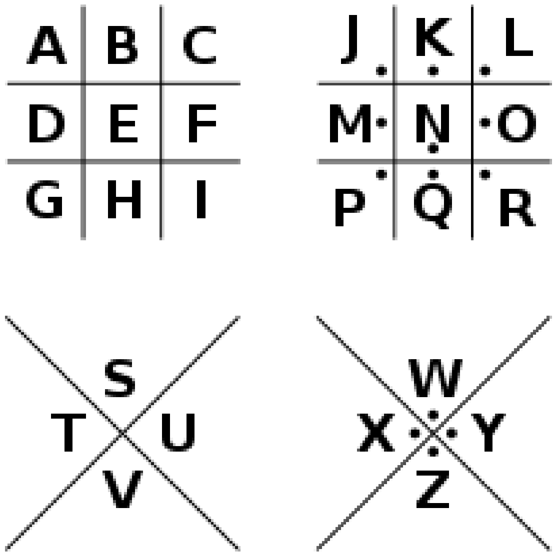
The following shows the word “Pigpen cipher” encrypted using the Pigpen cipher:



Note 8: Vigenère Table Printout

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
A	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
B	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A
C	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B
D	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C
E	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D
F	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E
G	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F
H	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G
I	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H
J	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I
K	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J
L	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K
M	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L
N	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M
O	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N
P	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
Q	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
R	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q
S	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
T	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
U	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T
V	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U
W	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	D
X	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	C
Y	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	B
Z	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y

Note 9: Pigpen Cipher Printout



Note 10: Solutions for the Kahoot Quiz (link available in additional resources)

Q1. Decode this message using a Caesar cipher (link available in additional resources"=) with a shift of 3: "U JXOHP QEB PMLQ"

X marks the spot

Q2. Using the keyword "KEY" encode this message using the Vigenère cipher: "HEADS UP"

RIYNW SZ

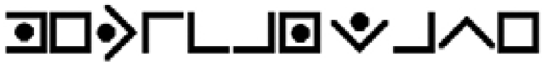
Q3. Which cipher was described as "le chiffre indechiffrable"?

The Vigenère cipher

Q4. Encode this message using a Caesar cipher with a shift of 10: "See You Later"

COO IYE VKDOB

Q5. Decode this message using the Pigpen cipher:



Mexican Wave

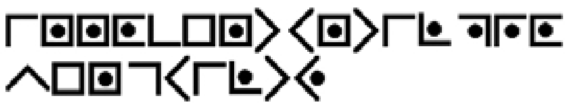
Q6. The word "Cryptography" has its origins in which language?

Greek

Q7. Using the keyword "VIGENERE" decode this message using the Vigenère cipher: "WGWGRPBYNWHYR"

Rocket Science

Q8. Decode this message using the Pigpen cipher:



Innocent Until Proven Guilty

Q9. What shift encodes "Good as Gold" as "XFFU RJ XFCU" using the Caesar cipher?

17

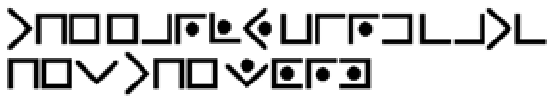
Q10. What shift did Julius Caesar use for his messages?

3

Q11. Using the keyword "LIMBO" encode this message using the Vigenère cipher: "Talk the Talk"

EIXL HSM FBZV

Q12. Encode this message using the Pigpen cipher: "Early Bird Catches the Worm".

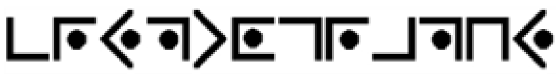


Q13. Which person is most well known for cracking the Enigma code during World War 2?

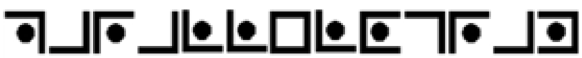
Alan Turing

Q14. Encode the phrase "Maths Sparks" using the Pigpen cipher.

Q15. Encode the word "Cryptography" using the Pigpen cipher.



Q16. Encode the word "Parallelogram" using the Pigpen cipher.



Sources and Additional Resources

https://www.youtube.com/watch?v=sMOZf4GN3oc (Caesar cipher video)

https://www.youtube.com/watch?v=s5XRTcLYy40 (Pigpen cipher video)

https://v2.cryptii.com/text/pigpen (Pigpen cipher translator)

https://play.kahoot.it/#/k/81f42961-54b0-4567-99be-381f2bf39f48 (Kahoot link)

Cryptography: Activity 1

Q1. Chris decides he wants to encrypt the phrase "ATTACK AT DAWN" using a Caesar cipher and a shift of 10. Follow the steps labelled 1 to 4 below to encrypt this message. One letter has been completed for you.

1. Fill in the numbers corresponding to each letter.
2. Add the shift (here it's 10) to each number.
3. Reduce each number modulo 26.
4. Read off the letters corresponding to the reduced numbers.

	A	T	T	A	C	K	A	T	D	A	W	N
I		19										
II		29										
III		3										
IV		D										

Q2. Sally wants to encrypt the phrase "BRUTE FORCE ATTACK" by a shift of 5.

1. Match the inner and outer wheels to begin (make sure A corresponds to A).
2. Turn the inner wheel clockwise by your shift (in this case 5).
3. For each letter in your message, find it on the inner wheel and write the corresponding letter on the outer wheel in the box below.

B	R	U	T	E	F	O	R	C	E	A	T	T	A	C	K

Q3. Mohammed wants to decode "VJCQB RB ODW" using a shift of 9. Using your wheel, can you decrypt the message?

V	J	C	Q	B	R	B	O	D	W

Cryptography: Activity 2

Q1. Using the keyword "CIPHER", encrypt the phrase "OVER AND OUT" by following the steps below:

I. Fill in the keyword letter by letter under the phrase, repeating if necessary.

II. Using the Vigenère table, find the letter where the two letters above meet (for the first letter, for example, we need to find where column O and row C coincide).

	O	V	E	R	A	N	D	O	U	T
I	C	I								
II	Q			Y						

Q2. Decrypt the phrase "FIEQ BYOY FROL" using the keyword "MONDAY" by following the steps below:

I. Fill in the keyword letter by letter under the phrase, repeating if necessary.

II. Find the letter of the keyword you are looking for along the top row of the Vigenère table. Then, follow down the column until you find the corresponding letter of the coded phrase. Read across to the first column to find the original letter.

	F	I	E	Q	B	Y	O	Y	F	R	O	L
I	M											
II	T											

Q3. Decrypt the phrase "UT BZ JMIGPFS" using the keyword "MATHS".

	U	T	B	Z	J	M	I	G	P	F	S
I											
II											

Invariants

Introduction

Our brains have an amazing capacity to recognise objects as being the same even after they have been flipped, rotated, moved or otherwise distorted. Similarly, in mathematics a quantity or relationship can remain unchanged after a mathematical operation or transformation is applied. This idea is known as invariance and is rooted in many areas of mathematics including algebra, geometry and trigonometry, in addition to computer science. Angles, for example, are invariant under the operations of rotation and reflection. In other words, the angle will still be the same when a rotation or reflection is applied. The ability to recognise such invariants and to decompose problems in mathematics are important skills which will be explored in this workshop.

Aim of Workshop

The aim of this workshop is to introduce students to the concept of invariants through a series of puzzles. Students will also be encouraged to decompose the problems as a series of variables and to identify the invariant.

Learning Outcomes

By the end of this workshop, students will be able to:

- Describe, in their own words, what is meant by invariance
- Solve problems involving invariants

Materials and Resources

Per Group: Scissors, Sellotape, nets of shapes, 7 cups, a bar of chocolate (different numbers of squares are optional)

KEY WORDS

Parity

The state of a number being even or odd

Invariant

An expression or quantity which remains unchanged after a mathematical operation or transformation is applied.

Invariants: Workshop Outline

Suggested Time (Total mins)	Activity	Description
5 mins (00:05)	Introduction to Invariance	· Introduce the concept of invariance and provide examples (see **Workshop Introduction**) · Compare an invariant to a variable
15 mins (00:20)	Activity 1 Nets and Cubes	· Hand out the nets, Sellotape and scissors to each group (see **Appendix – Note 1**) · **Activity 1:** Students construct the shapes using Sellotape and try to identify the least number of cuts to turn the shapes back to nets again (see **Appendix – Note 2**) · Encourage students to compare results with each other. Is there a relationship between the shapes and the number of cuts?
15 mins (00:35)	Activity 2 Cups	· Explain the rules of the cup game (see **Appendix – Note 3**) · **Activity 2:** In pairs or small groups, students try to find the initial states where it is possible to turn all the cups the right way up (see **Appendix – Note 4**)
15 mins (00:50)	Activity 3 Handshake Game	· Explain the rules of the handshake game (see **Appendix – Note 5**) · **Activity 3:** Students shake hands with each other for 30 seconds and count how many hands they shook in total. Why is there an even number of people with odd handshakes? · Demonstrate the results to the class (see **Appendix – Note 6**)
10 mins (01:00)	Activity 4 All the Chocolate	· Explain the rules of the game (see **Appendix – Note 7**) · **Activity 4:** In groups, students try to determine the number of cuts it takes to break a chocolate bar into individual pieces (see **Appendix – Note 8**)

Invariants: Workshop Appendix

Note 1: Nets of Shapes

Students create a variety of three-dimensional shapes from the nets and try to determine the least number of cuts that are needed to turn the shapes back to nets again. You may wish to encourage the students to consider the different characteristics of the shape (i.e. faces, edges, and vertices) and see if they can identify any relationships. See **Activity 1** for net templates.

Note 2: Solution for Nets of Shapes

There is a relationship between the number of vertices (v) and the cuts (c) required to reduce the shape back to its net form. As the number of vertices increases so too does the number of cuts. This can be represented as $v \rightarrow v + 1, c \rightarrow c + 1$

This means that the difference between the vertices and cuts required is invariant. The minimum number of vertices to form a three-dimensional shape is four. For example, it takes three cuts to reduce a triangular based pyramid to its net. The difference between vertices can, therefore, be represented by $v - 1 = c$

Note 3: Seven Cups

Seven cups are placed in a line on the table with some cups positioned upside down and some the right way up. The students must move all the cups the right way up. However, the cups may not be turned individually; you are only allowed to turn any two cups simultaneously. Encourage students to consider from which initial states of the cups is it possible to turn all the cups the right way up.

Note 4: Solution for Seven Cups

This activity concerns the number of upward cups u.

There are three possible moves for the cups:

- Flip two downward facing cups up. This has the effect of $u \rightarrow u + 2$
- Flip two upward facing cups down: $u \rightarrow u - 2$
- Flip one cup from up to down, and one from down to up: $u \rightarrow u$

No matter what move we make, the parity of the upward cups is conserved. So, if u is initially even, there is no move we can do to make this odd. Consequently, u will never equal seven. If u is initially odd however, we can continue flipping downward cups until $u = 7$.

Note 5: Handshake Game

Students move around the classroom and shake hands for 30 seconds whilst ensuring to keep track of the number of handshakes. After 30 seconds, count the number of people with an odd number of handshakes. The number of people with odd handshakes should be even. Repeat the handshaking process again. The number of people with odd handshakes should still be even. Encourage students to consider why there is an even number of people with odd handshakes.

Note 6: Solution for Handshake Game

Ask the students with an even number of handshakes (e) to stand at one end of the table and all the students with an odd number of handshakes (o) to stand at the other end. Demonstrate the possible moves:

- Have two evens shake hands and instruct them to go to the other side of the table as their new number of handshakes is now odd:

  $e \rightarrow e - 2,\ o \rightarrow o + 2$

- Have the odds shake hands an instruct them to go to the other side:

  $e \rightarrow e + 2, o \rightarrow o - 2$

- Have an odd and an even shake hands and instruct them to swap sides:

  $e \rightarrow e,\ o \rightarrow o$

Highlight at the beginning that everyone starts with zero handshakes and thus we begin with an even number (0) of people with an odd number of handshakes. Since the number of handshakes can only change by two, the parity is conserved.

Note 7: All the Chocolate

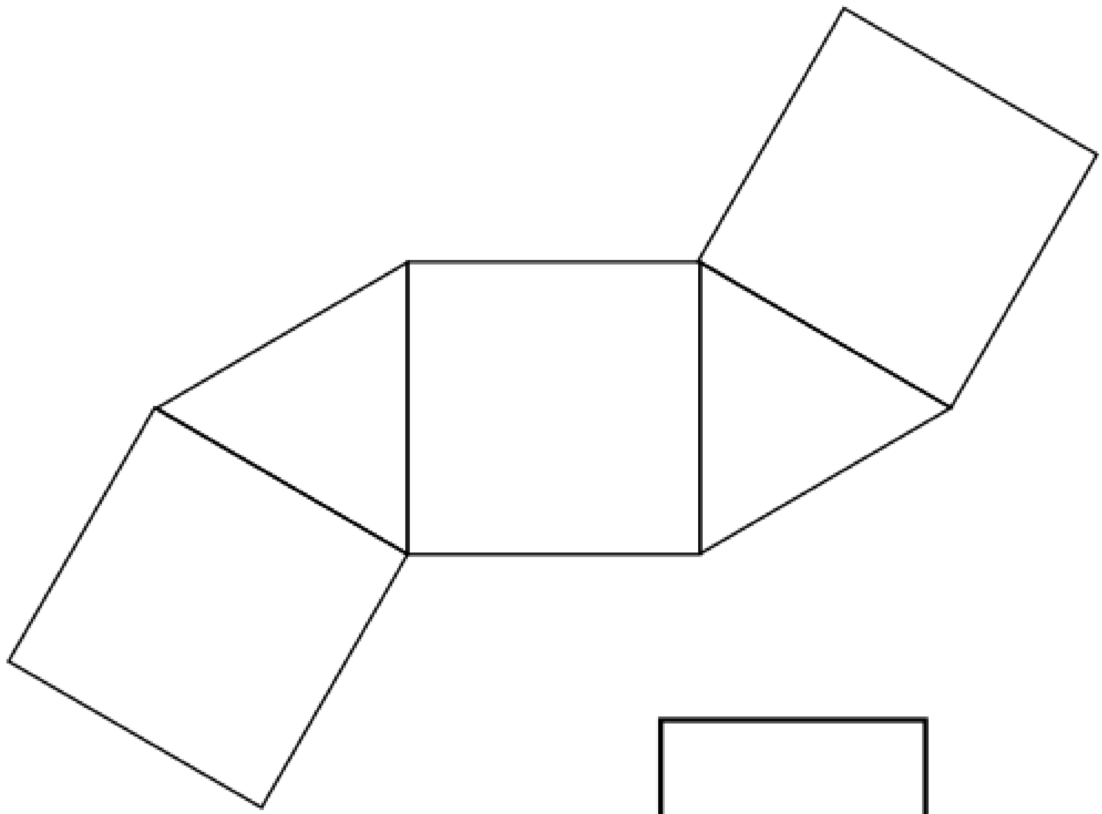
Pass around a chocolate bar with divisions, giving everyone a chance to make a cut along the lines of the divisions. Keep track of the number of cuts. How many cuts does it take to break a chocolate bar into individual pieces? Prompt the students to consider the possible moves.

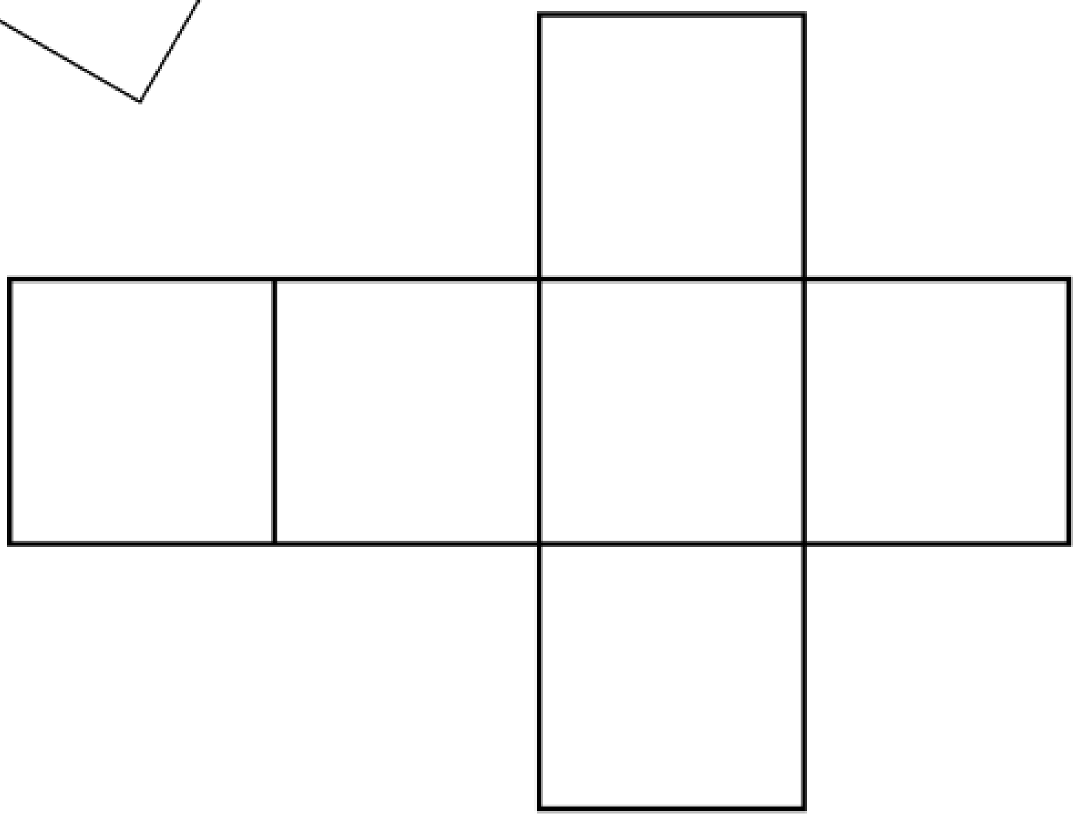
Note 8: Solution for All the Chocolate

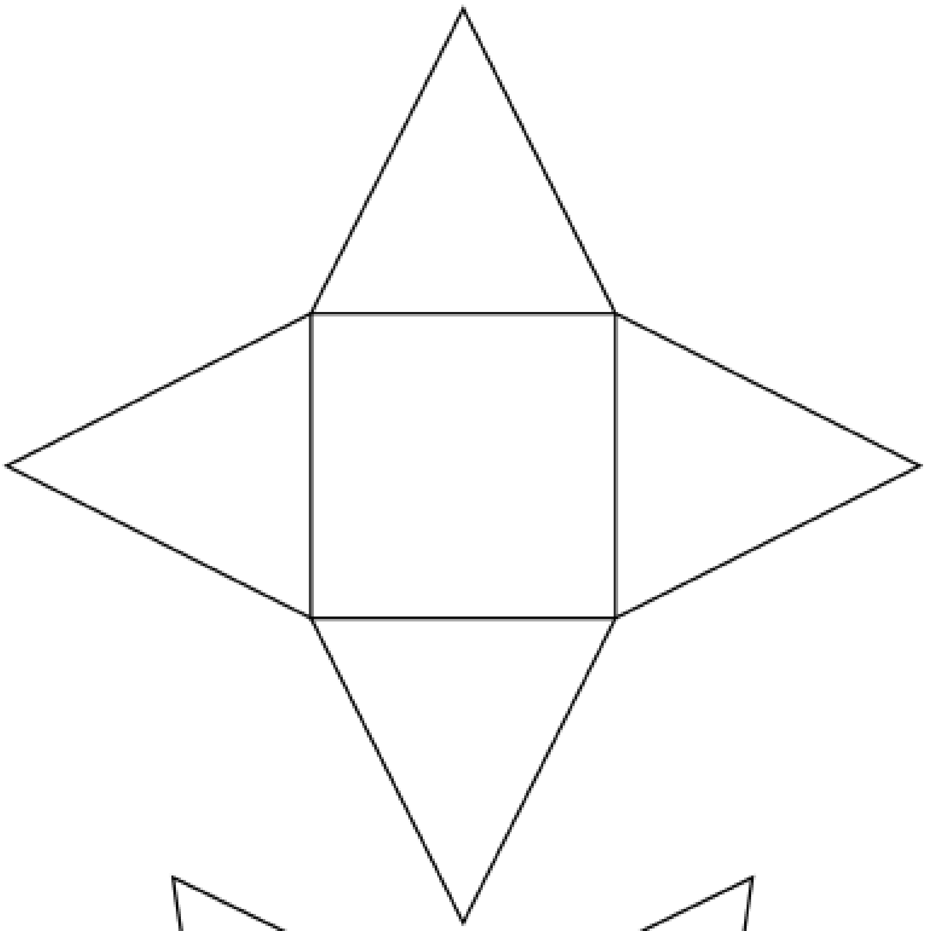
You begin with a single piece of chocolate. As you make a cut, you increase the number of pieces by one. This can be expressed as $n \rightarrow n + 1, c \rightarrow c + 1$

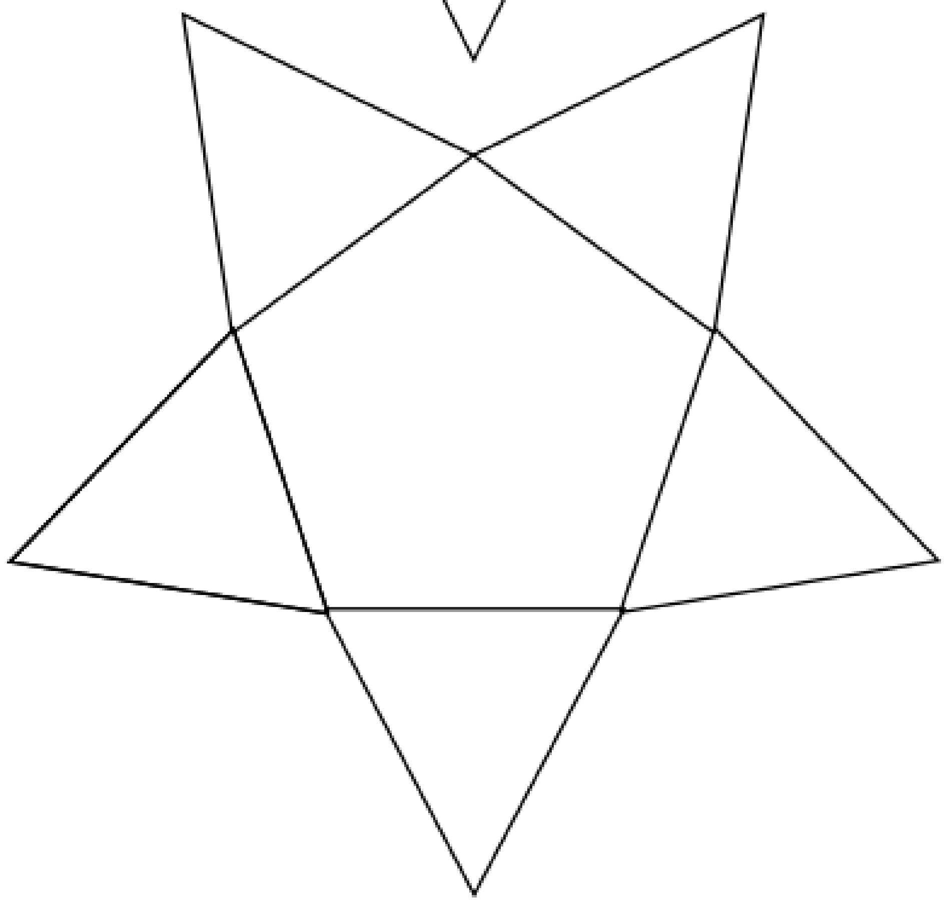
This means that the difference between the number of pieces and cuts required is invariant. We initially have one piece of chocolate and zero cuts, so the difference is one $n - 1 = c$. If you start with 16 pieces of chocolate, then it will take 15 cuts to break the bar into individual pieces.

Invariants: Activity 1









Force Vectors

Introduction

Everything from the paths around of Dublin to interstellar space is subject to different types of forces. Recognising such forces will allow us to better understand the interactions inside the nucleus of an atom, the construction of a bridge, or even the decay of a star into a supernova. The work of Sir Isaac Newton has played a vital role in advancing our knowledge of forces and his three laws underpin much of modern-day physics and mathematics. In this workshop, we will explore Newton's first law, which states that an object will remain at rest unless acted upon by an unbalanced force.

Aim of Workshop

The aim of the workshop is to introduce students to force vectors in order to show them the applications of physics in the world around us. Students will also use their knowledge of algebra and trigonometry to examine the mathematical and physical significance of force vectors.

Learning Outcomes

By the end of this workshop, students will be able to:

- Describe, in their own words, what is meant by a net force
- Draw unbalanced and balanced forces
- Solve problems related to force vectors

KEY WORDS

Force Vector

A quantity that describes both the magnitude and the direction of a force

Net Force

The sum of all forces acting on an object

Force Vectors: Workshop Outline

Suggested Time (Total mins)	Activity	Description
5 mins (00:05)	Introduction to Force	· Introduce the concept of a force and refer to Newton's second law (see **Appendix – Note 1**)
15 mins (00:20)	Activity 1 Net force	· Explain what it meant by a force vector (see **Key Words**) · Give examples of balanced and unbalanced forces (see **Appendix – Note 2**) and refer to the net force (see **Key Words**) · **Activity Sheet 1**: Students calculate the net force for the different boxes (see **Appendix – Note 3**)
15 mins (00:35)	Activity 2 Vector Components	· Explain the horizontal and vertical component of a force vector and provide an example (see **Appendix – Note 4**) · **Activity Sheet 2:** Students attempt to find the vector components acting on the different boxes (see **Appendix – Note 5**)
15 mins (00:50)	Activity 3 The Stubborn Donkey	· **Activity Sheet 3:** Students calculate the force the donkey is exerting on the ropes (see **Appendix – Note 6**)

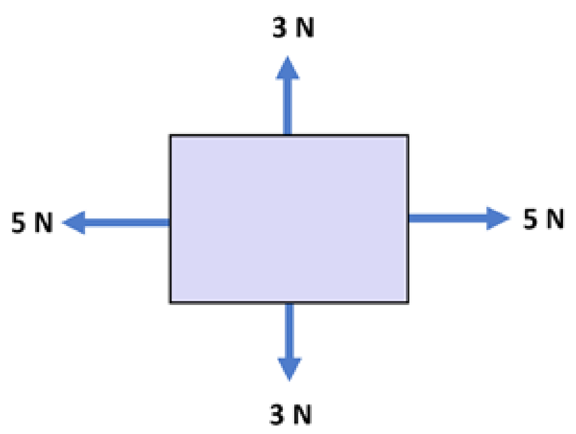
Force Vectors: Workshop Appendix

Note 1: Force

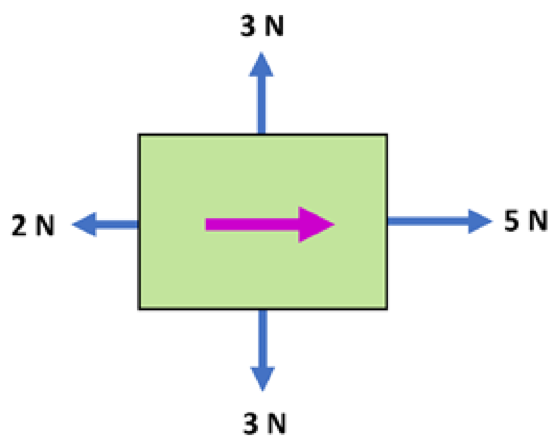
Force is any interaction that, when unopposed, causes a change in the motion of an object. This idea is summarised by Sir Isaac Newton in his Laws of Motion. For example, Newton's second law describes force as the mass of an object times its acceleration, more commonly denoted by the formula, F=ma. The SI unit for force is Newton (N), with 1 Newton equal to 1 kg × m/s^2

Note 2: Balanced and Unbalanced Forces

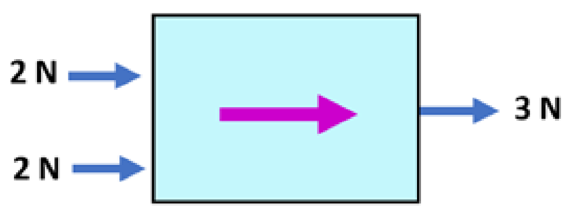
The table below shows examples of balanced and unbalanced forces. Note: the pink arrow in each case represents the force vector.



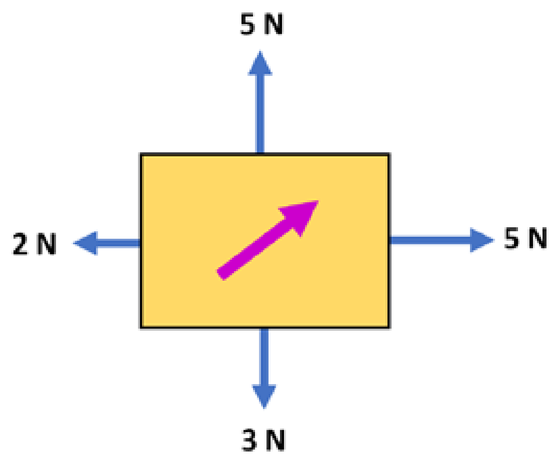
The forces above are said to be balanced as they 'cancel' each other out. There is thus a net force of 0 N and the box will remain stationary



The forces are unbalanced in the horizontal direction as there is a greater pulling force to the right. The net force is thus 3 N to the right



The forces are unbalanced, and the net force will be 7 N to the right since there is a pushing force of 4 N to the right in addition to a pulling force of 3 N to the right.



In this example, the forces are unbalanced in both the horizontal and vertical direction, and thus the box will move in a northeast direction.

Note 3: Solutions for Activity 1

Calculate the net force acting on each of the boxes below. Make sure to include the direction of the net force (i.e. left or right)

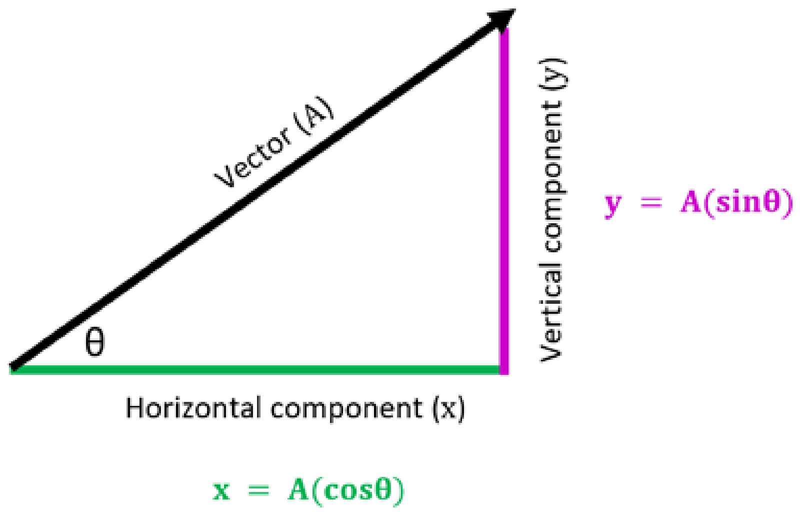
(a) 3 N right	(b) 5 N right
(c) 0 N	(d) 10 N right
(e) 6 N right	(f) 3 N right
(g) 3 N left	(h) 0 N

Note 4: Vector Components

A vector acting on an object at an angle θ can be broken down into its horizontal (x-axis) and vertical (y-axis) components as shown below. Since it is a right-angled triangle, we can find x and y using the trigonometric functions (Note: you may wish to revise this with the students).

$$\sin\theta = \frac{y}{A}$$
$$y = A(\sin\theta)$$

$$\cos\theta = \frac{x}{A}$$
$$x = A(\cos\theta)$$



Example 1: Find the x and y-components acting on the box below

Draw in the x and y-axes first and find the relevant angle.

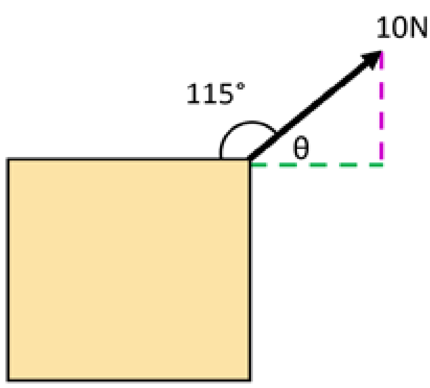
Angle $\theta = 180° - 115° = 65°$

$x = 10\ (\cos 65°)$

$x = 4.22$ N

$y = 10(\sin 65°)$

$y = 9.06$ N



Example 2: Find the x and y-components acting on the box below

Draw in the x and y-axes first and find the relevant angle.

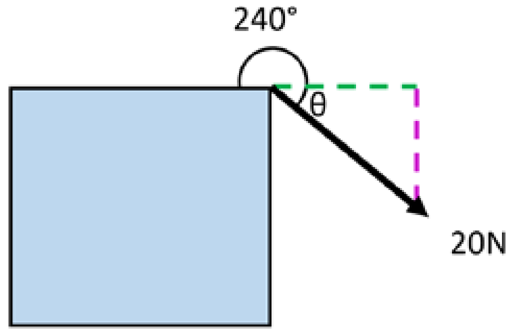
Angle $\theta = 240° - 180° = 60°$

$x = 20\ (\cos 60°)$

$x = 10\ N$

$y = -20\ (\sin 60°)$

$y = -17.32\ N$



(y is negative since the vector is downward pointing. Consider the origin of the vector to be positioned at (0,0). We can thus identify the quadrants where x and y are positive and negative)

Note 5: Solutions for Activity 2

Find the vector components acting on the boxes below (Hint: Start by drawing the x and y-axes and think about which angle you need to use).

(a) x-component: $20\sqrt{2}\ N$ y-component: $20\sqrt{2}\ N$

(b) x-component: 24.62 N y-component: 4.34 N

(c) x-component: 58.91 N y-component: -27.47 N

(d) x-component: 49.24 N y-component: -8.68 N

Note 6: Solutions for Activity 3

Q1. Two people are struggling to move their donkey as shown in the diagram below. Calculate the x and y-components for the force exerted by person A and by person B (i.e. the forces acting on the donkey).

Person A:

$x = -80(\cos 75°)$ $y = 80(\sin 75°)$

$x = -20.71N$ (i.e. negative x-direction) $y = 77.27N$

Person B:

$x = 120(\cos 60°)$ $y = 120(\sin 60°)$

$x = 60N$ $y = 103.92N$

Q2. Now calculate the net force acting in the x and y-directions based on the forces exerted by person A and person B only.

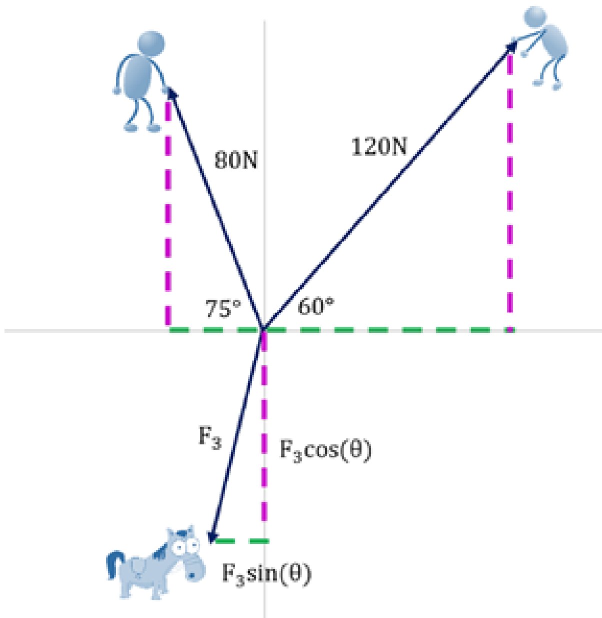
Person A is exerting a force of 77.27N in the y-direction and Person B is exerting a force of 103.92N in the y-direction. The net force is thus 181.19N in the positive y-direction.

Person A is exerting a force of -20.71N in the x-direction (see axes) and Person B is exerting a force of 60N in the x-direction. The net force is thus 39.29N in the positive x-direction.

Q3. Hence calculate the x and y-components for the force the donkey must be exerting on the ropes to make the net force = 0 in both the x and y-direction.

Let $F_3 \sin(\theta)$ be the x-component and $F_3 \cos(\theta)$ be the y-component for the force the donkey is exerting on the ropes

Since the people are unable to move their donkey, the net forces must equal 0 in both the x and y-directions. We thus have:



$F_3 \cos(\theta) + 77.27\text{N} = 0$

$\Rightarrow F_3 \cos(\theta) = -77.27\text{N}$ (y-component)

$F_3 \sin(\theta) + 39.29\text{N} = 0$

$\Rightarrow F_3 \sin(\theta) = -39.29\text{N}$ (x-component)

Q4. Using Pythagoras' theorem, can you now find the force F_3 that the donkey is exerting on the ropes?

$(F_3)^2 = (-77.27\text{N})^2 + (-39.29)^2$

$(F_3)^2 = 7514.357$

$F_3 = 86.69\text{N}$

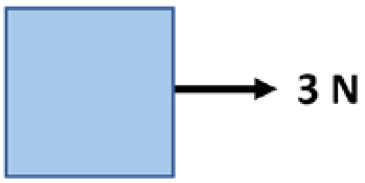
Sources and Additional Resources

https://physics.tutorvista.com/motion/force-vectors.html (Vector forces)

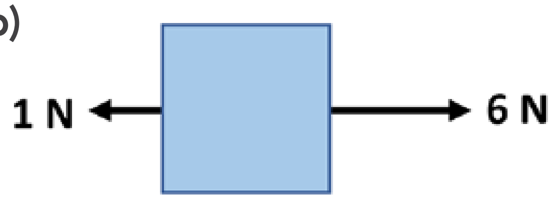
https://www.physicsclassroom.com/class/vectors/Lesson-3/Resolution-of-Forces (Vector components)

Force Vectors: Activity 1

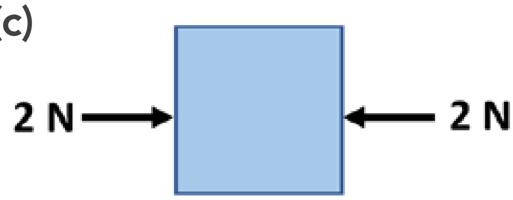
Calculate the net force acting on each of the boxes below. Make sure to include the direction of the net force (i.e. left or right)

(a)


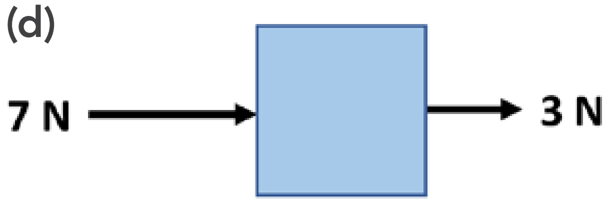
Net force: _____________

(b)


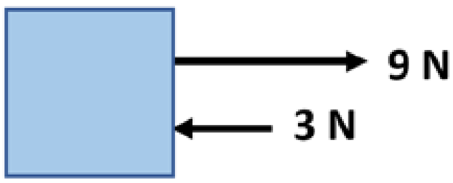
Net force: _____________

(c)


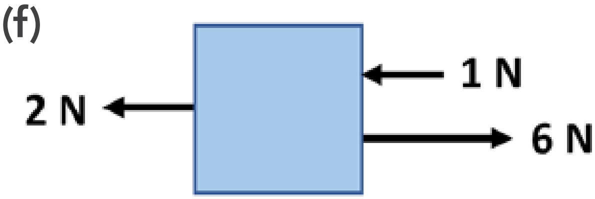
Net force: _____________

(d)


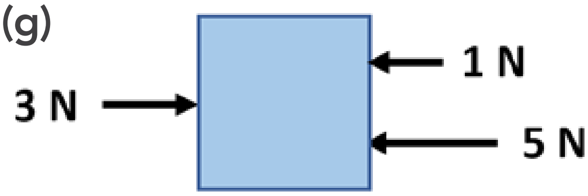
Net force: _____________

(e)


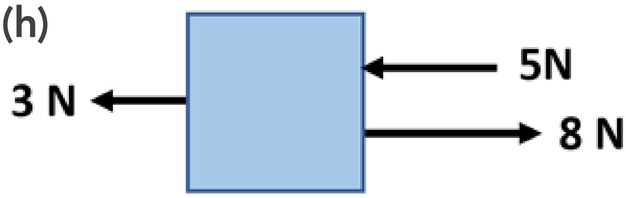
Net force: _____________

(f)


Net force: _____________

(g)


Net force: _____________

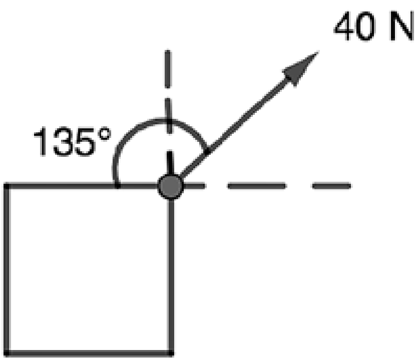
(h)


Net force: _____________

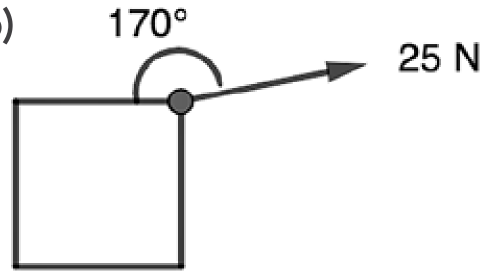
Force Vectors: Activity 2

Find the vector components acting on the boxes below (Hint: Start by drawing the x and y-axes and think about which angle you need to use).

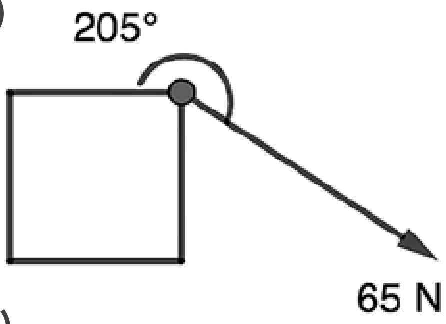
(a)



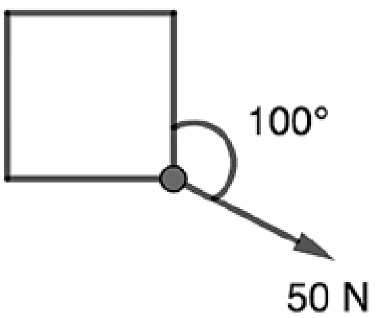
(b)



(c)

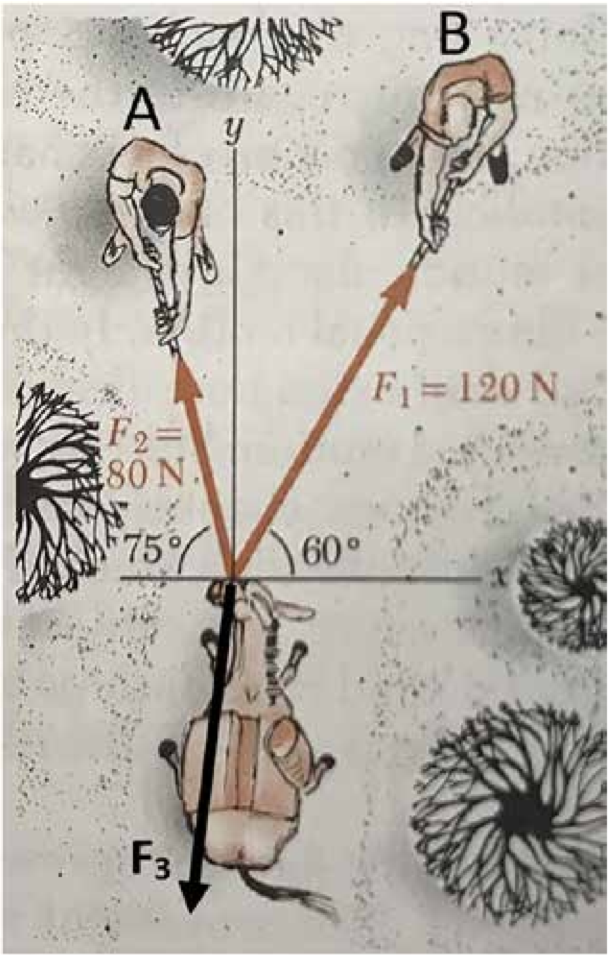


(d)



Force Vectors: Activity 3

Q1. Two people are struggling to move their donkey as shown in the diagram below. Calculate the x and y-components for the force exerted by person A and by person B (i.e. the forces acting on the donkey). (Note: be careful with the signs – think about the x and y-axes).



Q2. Now calculate the net force acting in the x and y-directions based on the forces exerted by person A and person B only. You may wish to draw force vectors to help you.

Q3. Hence calculate the x and y-components for the force the donkey must be exerting on the ropes to make the net force = 0 in both the x and y-direction (Note: let F_3 be the magnitude of the force the donkey is exerting on the ropes and θ be the angle of the vector).

Q4. Using Pythagoras' theorem, can you now find the force (F_3) that the donkey is exerting on the ropes?

Logic Machine

Introduction

In 1866, William Stanley Jevons, an English logician and economist constructed a machine known as the 'logical piano', which was capable of solving complicated problems with superhuman speed. This machine was inspired by George Boole's work on mathematical logic, known as 'Boolean Algebra'. Interestingly, a number of researchers in the 1930s noticed that the binary numbers (0 and 1), combined with the Boolean values of True and False, could be used to construct electrical switching circuits and thus used to design electronic computers. In this workshop we will introduce a variant of Jevons' logical piano, popularised by Martin Gardner in the mid-20th century, known as the 'the logical punch cards'. These punch cards combined logic with binary arithmetic and were widely used in the early computing era as they provided a convenient solution for storage, input and output for a computer.

Aim of Workshop

The aim of this workshop is to introduce students to the binary system and its connections to logic. The use of the logical punch cards will serve to demonstrate these principles in action and provide students with some historical context so that they can better appreciate the mathematics, ingenuity, and history behind a modern computer.

Learning Outcomes

By the end of this workshop, students will be able to:

- Construct a truth table to solve problems
- Explain, in their own words, how logical punch cards work
- Solve deductive reasoning problems using punch cards

Materials and Resources

A deck of 32 punch cards for each group of students (see appendix for template), activity sheets.

KEY WORDS

Binary
A system of counting which has '2' as its base rather than '10'. It is also commonly referred to as 'base 2'

Logic
The systematic study of the form of argument

Premise
A previous statement from which another follows as a conclusion

Logic Machine: Workshop Outline

Suggested Time (Total mins)	Activity	Description
10 mins (00:10)	Introduction to binary and logic	· Introduce the binary system and mention its importance in computer science (see **Appendix – Note 1)** · Draw connections between binary and Boolean algebra (see **Appendix – Note 2**) · Outline the historical background of Boolean algebra
15 mins (00:25)	Activity 1 Party invitations	· Introduce the concept of truth tables (see **Appendix – Note 3**) · **Activity Sheet 1:** Students complete Activity 1 using a truth table (see **Appendix – Note 4**)
10 mins (00:35)	Punch cards	· Demonstrate how to use punch cards to solve problems (see **Appendix – Note 5**)
15 mins (00:50)	Activity 2 Who is watching TV?	· **Activity Sheet 2:** In pairs or groups of three, students try to solve who is watching TV using the punch cards (see **Appendix – Note 6)**
10 mins (01:00)	Hamilton and Jevons	· Outline the work of Margaret Hamilton in the Apollo 11 mission (see **Appendix – Note 7**) · Mention William Stanley Jevons & his 'Logic piano' (see **Appendix – Note 8)**
10 mins (01:10)	Activity 3 Similar triangles (Optional)	· Explain the premises for the similar triangles puzzle (see **Appendix – Note 9**) · **Activity Sheet 3:** In pairs or groups of three, students attempt to solve the similar triangle puzzle using punch cards (see **Appendix – Note 10**)

Logic Machine: Workshop Appendix

Note 1: Introduction to Binary

Binary is a counting system based on successive powers of 2 and is made up of only two digits: 0 and 1. Gottfried Wilhelm Leibniz, a German mathematician, was the first person who studied the binary system in depth during the 17th century, solely for mathematical interest. Nowadays, however, binary is widely used in computer programming as it is the most efficient way to operate logic circuits. This is due to the fact that the digits 0 and 1 represent OFF and ON respectively and can, therefore, be used to control the state of electrical signals in a circuit.

Note: For more information on binary please refer to 'Base Systems', Maths Sparks Volume I.

Note 2: Introduction to Logic

Logic is an important concept that has applications in a wide variety of disciplines including computer science, linguistics, philosophy and mathematics. Whilst there are many different types of logic, this workshop will refer to Aristotelian logic, which concerns deductive reasoning as expressed in syllogisms e.g. All Greeks are men, all men are mortal, therefore all Greeks are mortal. However, we also draw on the work of George Boole, an English mathematician and first professor of mathematics at Queen's College, Cork (now UCC), who extended Aristotle's philosophical approach to logic and developed a mathematical system for interpreting logical statements. This later became known as 'Boolean algebra' and can be used to solve certain problems involving deductive reasoning.

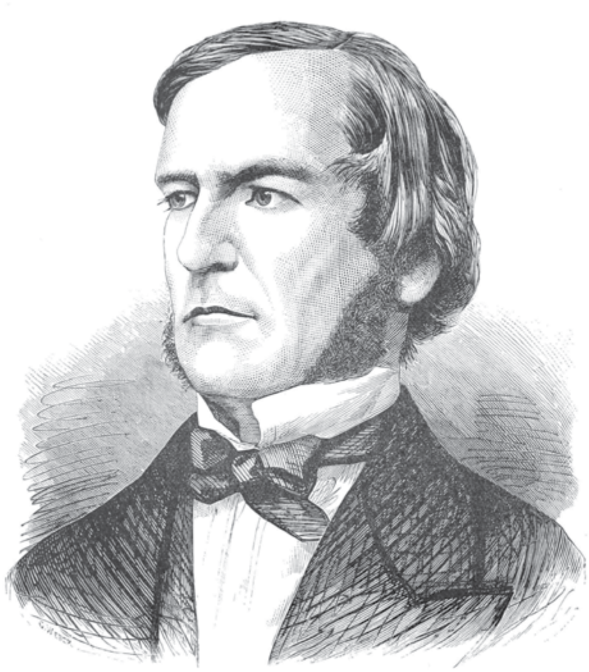


Figure 2: George Boole

Whilst the Greeks studied logic in the form of statements, Boolean algebra used the variables 'true' and 'false', which are commonly denoted by the binary digits '1' and '0' respectively. With this, Boole was able to condense a string of statements into a binary sequence. It is this connection between logic and binary that makes Boolean algebra so powerful in the digital world.

Note: For more information on logic please refer to 'Logic Gates', Maths Sparks Volume II.

Note 3: Truth Tables

A truth table is a convenient way of describing the outcomes of basic logic operations. They comprise a column for each variable, and rows that represent all possible situations (i.e. true and false combinations of our variables). We will use **Activity Sheet 1** to demonstrate how a truth table can be used to solve problems (see **Note 4**).

Note 4: Solutions for Activity 1

You have three friends: John, Paul, and Mary, whom you would like to invite to your party

1. If you invite Mary, you must invite John
2. If you don't invite Paul, you must invite both John and Mary
3. You must invite either John or Mary, but not both

Q1. Fill in the last two columns of the truth table below:

Row	John	Paul	Mary	Premise 1	Premise 2	Premise 3
i.	0	0	0	✔	X	X
ii.	0	0	1	X	X	✔
iii.	0	1	0	✔	✔	X
iv.	0	1	1	X	✔	✔
v.	1	0	0	✔	X	✔
vi.	1	0	1	✔	✔	X
vii.	1	1	0	✔	✔	✔
viii.	1	1	1	✔	✔	X

1 = invite, 0 = does not invite

Note: Each row in the table represents one scenario of who we invite to the party.

For example, row (i) represents all three people are not invited whereas row (iv) represents a situation where Paul and Mary are invited but John is not etc.

Since there are two choices for each person (invited or not invited), the total number of possible scenarios is $2 \times 2 \times 2 = 2^3 = 8$.

We can formulate the total number of combinations as follows:

Total number of combinations = 2^x, where x is the number of variables.

Since the truth table exhausts all possible scenarios, we can use it to eliminate combinations that do not satisfy our premises as illustrated above.

Q2. Based on the table above, who should you invite to the party?

You should invite John and Paul to the party but not Mary since this is the only combination that follows all three premises (see row (vii))

Note 5: Using Punch Cards

The method of using truth tables can become rather tedious as the number of variables increase, which, in turn, will also increase the number of possible combinations. This is where the punch card comes in (see Figure 3). The punch card works under the same principle as the truth table, with cards representing each of the rows of the truth table. The punch cards provide a better method of filtering and removing contradictory combinations and save time when we have a large number of variables. For **Activity Sheet 1,** we would use 8 individual punch cards with the last three slots on each card denoting the three variables (i.e. John, Paul and Mary).

In order to demonstrate how punch cards work, we will use the same example as outlined in **Activity Sheet 1**. However, we will use the following abbreviations to simplify our explanation:

$\mathrm{J} = \text{John is invited}$

$\bar{\mathrm{J}} = \text{John is not invited (read as "J bar")}$

$\mathrm{P} = \text{Paul is invited}$

$\bar{\mathrm{P}} = \text{Paul is not invited (read as "P bar")}$

$\mathrm{M} = \text{Mary is invited}$

$\bar{\mathrm{M}} = \text{Mary is not invited (read as "M bar")}$

Figure 3: Example of a punch card

Premise 1: If you invite Mary, you must invite John

Based on the first premise, we are looking to eliminate combinations where Mary is invited but John is not invited, i.e. eliminate combinations with $\bar{\mathrm{J}}\,\mathrm{M}$. As shown in the truth table, we can eliminate row (ii) and row (iv) which do not satisfy the first premise. The converse, however, where John is invited but Mary is not invited might or might not be true, so we can keep those combinations.

	John is invited	Paul is invited	Mary is invited	Keep/discard
i.	0	0	0	Keep
ii.	0	0	1	Discard
iii.	0	1	0	Keep
iv.	0	1	1	Discard
v.	1	0	0	Keep
vi.	1	0	1	Keep
vii.	1	1	0	Keep
viii.	1	1	1	Keep

To eliminate the contradictory combinations (row (ii) and (iv)) using the punch cards, we first insert a pen (or skewer) into the slot labelled 'Mary' and lift the cards. As the cards that correspond to M (Mary is invited) have an open slit, they will not be lifted. Whereas cards that correspond to $\bar{\mathrm{M}}$ (Mary is not invited) will not have an open slit, hence they will be lifted and stay on the pen (see Figure 4). The deck is now sorted into 2 piles of cards:

Pile $\bar{\mathrm{M}}$	Mary is not invited
Pile M	Mary is invited

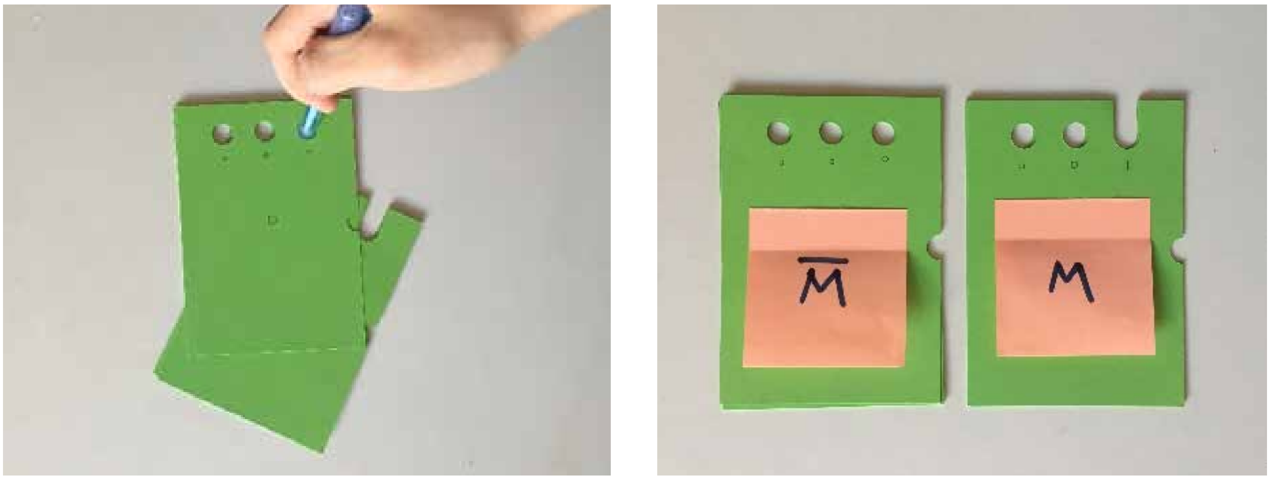


Figure 4: Filtering combinations where Mary is invited

From Pile M, we need to further separate combinations into piles where John is not invited, and John is invited. We achieve this by inserting a pen into the slot labelled John and lifting the cards. As before, the cards without a slit will be lifted and the rest will be left behind. This will result in two more piles:

Pile $\bar{\mathrm{M}}$	Mary is not invited
Pile $\bar{\mathrm{J}}\mathrm{M}$	John is not invited but Mary is invited
Pile JM	John and Mary both not invited

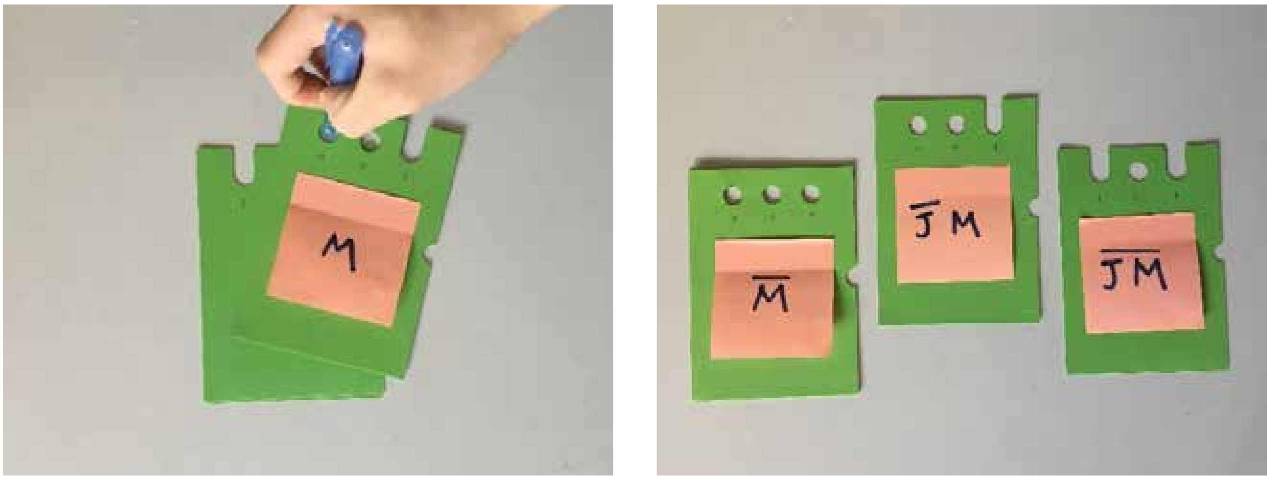


Figure 5: Filtering combinations where Mary and John are invited

We discard Pile $\overline{J}M$ because this pile does not satisfy premise 1. We can now combine Pile $\overline{M}$ with Pile JM and proceed to the next premise (Note: the discarded cards are no longer needed).

Premise 2: If you don't invite Paul, you must invite both John and Mary.

As the premise suggests, we only keep the combination where Paul is not invited but both John and Mary are invited (row vi). Since this premise does not concern what happens when Paul is invited, we can keep the rows with a 1 in the second column since they are still valid (i.e. row (iii), (iv), (vii), and (viii)). Note: the red or struck through lines signify the piles we discarded based on the first premise and are therefore no longer valid.

	John is invited	Paul is invited	Mary is invited	Keep/discard
i.	0	0	0	Keep
~~ii.~~	~~0~~	~~0~~	~~1~~	~~Discard~~
iii.	0	1	0	Keep
~~iv.~~	~~0~~	~~1~~	~~1~~	~~Discard~~
v.	1	0	0	Keep
vi.	1	0	1	Keep
vii.	1	1	0	Keep
viii.	1	1	1	Keep

Using the punch cards, we separate $\bar{\mathrm{P}}$ with P using a pen and the slot labelled P. We do not discard pile P since the premise says nothing about what happens if Paul is invited.
(Note: Since pile $\bar{\mathrm{P}}$ only contains three cards ($\overline{\mathrm{JPM}}$, $\mathrm{J}\overline{\mathrm{PM}}$ and $\mathrm{J}\bar{\mathrm{P}}\mathrm{M}$), it is possible to go through all three cards and select the one that does not contradict the premise ($\mathrm{J}\bar{\mathrm{P}}\,\mathrm{M}$). However, we will outline the formal approach below since this method would be tedious for a larger sample of punch cards).

Pile $\bar{\mathrm{P}}$	Paul is not invited
Pile P	Paul is invited

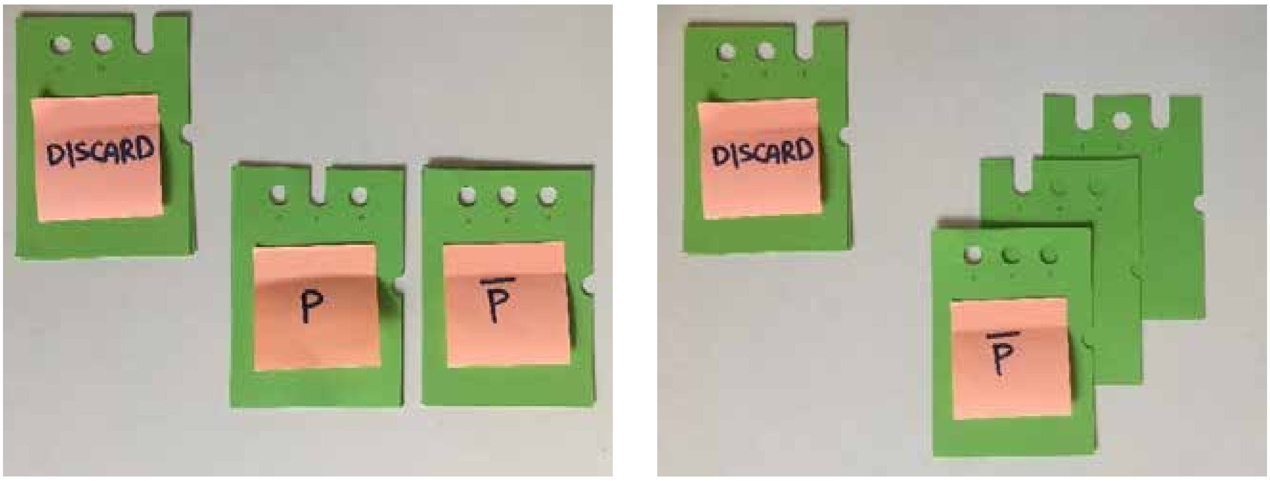


Figure 6: Filtering combinations where Paul is not invited

Once we have identified the pile $\bar{\mathrm{P}}$ (Paul is not invited), we further divide pile $\bar{\mathrm{P}}$ into two piles:

Pile $\mathrm{J}\bar{\mathrm{P}}$	John is invited when Paul is not invited
Pile $\overline{\mathrm{JP}}$	John is not invited when Paul is not invited

Figure 7: Filtering combinations where John is invited when Paul is not invited

Based on the second premise, we must invite John, so we can discard pile $\overline{JP}$. We then separate pile $J\overline{P}$ into two more piles:

Pile $J\overline{PM}$	John is invited but Mary is not invited when Paul is not invited
Pile $J\overline{P}M$	John and Mary are invited when Paul is not invited

We choose to keep $J\overline{P}M$ and discard $J\overline{PM}$ We can now recombine this card with the pile P as this premise does not say anything about situations where P is invited so this pile is still valid.

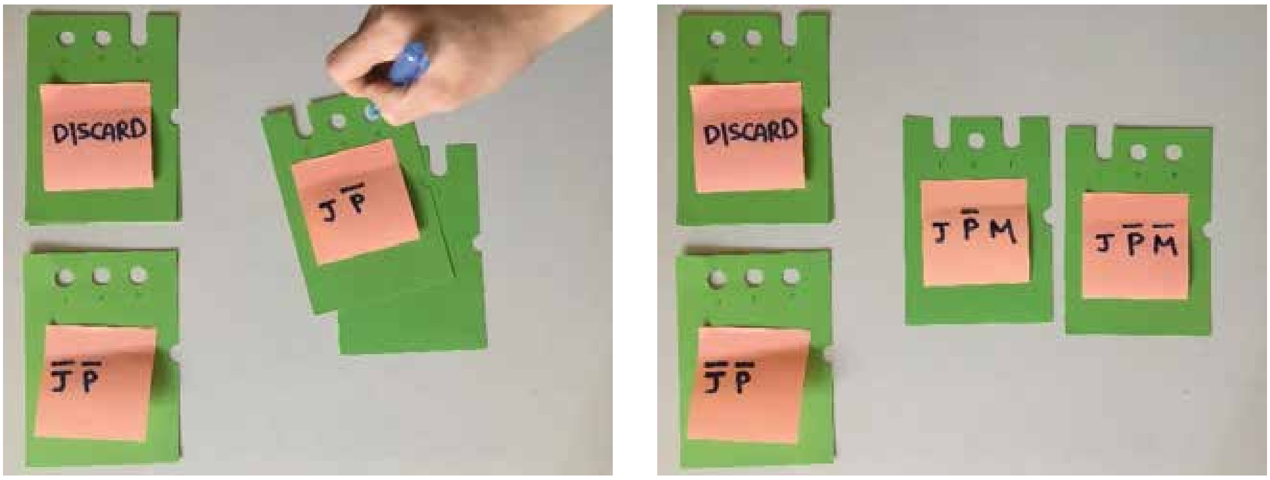


Figure 8: Filtering combinations where John and Mary are invited when Paul is not invited

Premise 3: You must invite either John or Mary, but not both.

Based on this final premise we want to eliminate combinations where John and Mary are both invited, i.e. eliminate JM. So, we discard row (vii) and (viii). Since, the term 'either' was used, it means that we should at least invite one of John and Mary. Hence, we can eliminate the combination where John and Mary are both uninvited, i.e. eliminate $\overline{JM}$. This leaves us with one final combination which brings us the answer that we should invite John and Paul. Note: the red or struck through rows in the table signify the piles we already discarded based on both the first and second premise and are therefore no longer in the pile.

	John is invited	Paul is invited	Mary is invited	Keep/discard
~~i.~~	~~0~~	~~0~~	~~0~~	~~Discard~~
ii.	~~0~~	~~0~~	~~1~~	~~Discard~~
iii.	0	1	0	Discard
~~iv.~~	~~0~~	~~1~~	~~1~~	~~Discard~~
~~v.~~	~~1~~	~~0~~	~~0~~	~~Discard~~
vi.	1	0	1	Discard
vii.	1	1	0	Keep
viii.	1	1	1	Discard

With the punch cards, we do this by first diving the deck into two piles using the first slot to obtain:

Pile $\bar{J}$	John is not invited
Pile J	John is invited

Then, from each of the piles, we separate M with $\overline{M}$ using the third slot to obtain four piles:

Pile $\bar{J}M$	John is not invited but Mary is invited
Pile $\overline{JM}$	John and Mary both not invited
Pile $J\overline{M}$	John is invited but Mary is not
Pile JM	John and Mary are both invited

Finally, we discard pile $\overline{J}\overline{M}$ and JM. At this point, there should be only one card left: $JP\overline{M}$, which is the combination that does not contradict any of the three premises above.

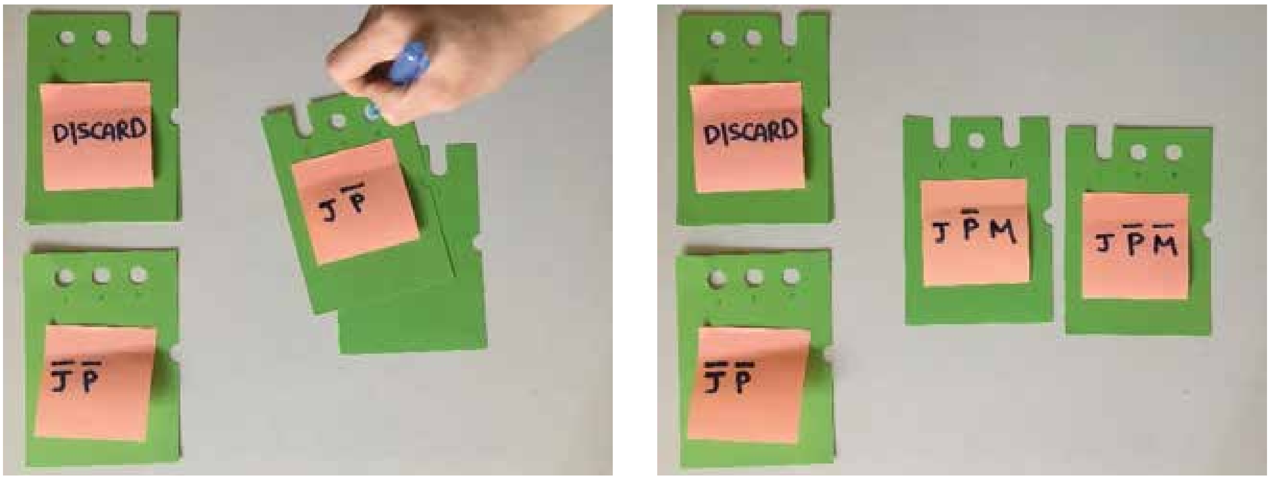


Figure 9: Filtering combinations where either John or Mary are invited but not both

Note: For a deck of 32 punch cards with 5 slots (instead of 3), you could divide the deck into 4 smaller decks: 0-7, 8-15, 16-23, and 24-31, based on the numbering on the cards and ignore the first two slots.

Note 6: Solutions for Activity 2

Who is watching television?

Albert and his wife Breda have three children: Ciara, David and Ellie.

1. If Albert is watching television, so is his wife.
2. Either David or Ellie, or both of them, are watching television.
3. Either Breda or Ciara, but not both, is watching television.
4. David and Ciara are either both watching television or both not watching television.
5. If Ellie is watching television, then Albert and David are also watching

Note: Each group of students will need the full set of 32 punch cards for this activity.

Q1. How many possible scenarios (combination of who is watching and who is not watching television) could we have for this TV watching session?

Since there are 5 people, each person could either be watching TV or not watching TV.

Thus, the number of possible scenarios is $2^5 = 32$

Q2. From the first premise, which combination should we eliminate?

AB – Albert and Breda both watching TV.

$\mathrm{A\bar{B}}$ – Albert is watching TV and Breda is not watching TV.

$\mathrm{\bar{A}B}$ – Breda is watching TV and Albert is not watching TV.

$\overline{\mathrm{AB}}$ – Albert and Breda both not watching TV.

The correct answer is $\mathrm{A\bar{B}}$. Students might also have $\mathrm{\bar{A}B}$ as an answer but the statement makes no assumption about what Albert will do if Breda watches television.

Q3. How could you remove the contradictory combination from Q2 from the deck? Describe Briefly.

- Using a set of 32 cards, label the five slots A, B, C, D and E respectively.
- To filter $\mathrm{A\bar{B}}$, we first separate cards of A with $\mathrm{\bar{A}}$ by inserting a pencil into the first slot and lifting the cards up. The cards that were left behind are the A pile (Albert is watching TV) while the cards that were lifted are the $\mathrm{\bar{A}}$ (Albert is not watching TV).
- From the A pile, we insert a pencil into the second slot and further separate the A pile into $\mathrm{A\bar{B}}$ pile (hang on the pencil) and AB pile (left behind).
- Discard the $\mathrm{A\bar{B}}$ pile and recombine the $\mathrm{\bar{A}}$ pile and AB pile. You should be left with 24 cards.

Q4. From the second premise, what combinations should we eliminate?

David is watching TV	Ellie is watching TV	Keep/discard
0	0	Discard
0	1	Keep
1	0	Keep
1	1	Keep

Based on the table above, we should eliminate the combination $\bar{D}\bar{E}$ (David and Ellie both are not watching TV). The combination $\bar{D}\bar{E}$ contradicts the premise because the premise uses the term 'either', therefore we need to have at least one of them (David or Ellie) watching TV.

Q5. How could you remove the contradictory combination from Q4 from the deck?

- To filter out $\bar{D}\bar{E}$, we first separate D from $\bar{D}$ using the fourth slot
- Then further divide the $\bar{D}$ pile into $\bar{D}E$ and $\bar{D}\bar{E}$ piles using the fifth slot, discard the $\bar{D}\bar{E}$ pile and recombine the rest. You should be left with 18 cards at this point

Q6. Based on the third premise, what combinations should we eliminate?

Breda is watching TV	Ciara is watching TV	Keep/discard
0	0	Discard
0	1	Keep
1	0	Keep
1	1	Discard

Based on the table above, there are two combinations that should be eliminated, $\bar{B}\bar{C}$ (Breda and Ciara both are not watching TV) and BC (Breda and Ciara both are watching TV).

Q7. How could you remove the contradictory combinations from Q6 from the deck?

Note: This premise involves removing two combinations, $\bar{B}\bar{C}$ and BC.

- We first separate B with $\bar{B}$ using the second slot
- Further divide both piles separately using the third slot to obtain four piles: BC, $B\bar{C}$, $\bar{B}C$, and $\bar{B}\bar{C}$. Discard the $\bar{B}\bar{C}$ and BC piles. Recombine the rest. You should be left with 9 cards

Q8. Based on the fourth premise, what combinations should we eliminate?

Ciara is watching TV	David is watching TV	Keep/discard
0	0	Keep
0	1	Discard
1	0	Discard
1	1	Keep

Based on the table above, the two combinations that should be eliminated are $C\bar{D}$ (Ciara is watching TV and David is not watching TV) and $\bar{C}D$ (David is watching TV and Ciara is not watching TV).

Q9. How could you remove the contradictory combinations from Q8 from the deck?

To remove the combination $C\bar{D}$ and $\bar{C}D$, we repeat a similar process as Q7, by first dividing the deck into C and $\bar{C}$ piles (using the third slot), then further divide both piles separately into CD, $\bar{C}D$, $C\bar{D}$ and $\bar{C}\bar{D}$ piles (using the fourth slot). Discard $C\bar{D}$ and $\bar{C}D$ piles and recombine the rest. You should be left with 4 cards at this point.

Q10. Based on the final premise, what combinations should we eliminate?

The premise only concerns cases where Ellie is watching TV, therefore we keep all the combinations with $\bar{E}$ (Ellie is not watching TV) and only focus on cases with E (Ellie is watching TV).

Ellie is watching TV	Albert is watching TV	David is watching TV	Keep/discard
1	0	0	Discard
1	0	1	Discard
1	1	0	Discard
1	1	1	Keep

Based on the table above, we need to discard three combinations: $E\bar{A}\bar{D}$, $EA\bar{D}$, and $E\bar{A}D$.

Q11. How could you remove the contradictory combinations from Q10 from the deck?

As there are only 4 cards left, it is possible to discard cards with the contradictory combinations by going through each of them. This will eliminate every card with the binary representation $0xx01$, $0xx11$, and $1xx01$.

Another method is to separate the deck into E and $\bar{E}$ pile (using the fifth slot). Since we are eliminating three combinations and only keeping one, it is faster to extract the combination we would like to keep: EAD.

You may notice that you cannot extract cards with combination EAD since these cards were removed by previous premises. Meaning that there are no cards in the $\bar{E}$ pile that satisfy the premise. You can then discard the $\bar{E}$ pile and you will be left with one card: $\bar{A}\bar{B}CD\bar{E}$.

Q12. Can you now determine who is watching TV?

Since the final punch card is $\bar{A}\bar{B}CD\bar{E}$, we now know that only Ciara and David are watching TV. This is the only combination/scenario that satisfies all five premises.

Note: It is possible to go through the premises in any order and with the cards shuffled; the result should be the same. Below is a table of cards that should be eliminated during each of the premises.

A	B	C	D	E	Premise 1	Premise 2	Premise 3	Premise 4	Premise 5
0	0	0	0	0		x	x		
0	0	0	0	1			x		x
0	0	0	1	0			x	x	
0	0	0	1	1			x	x	x
0	0	1	0	0		x		x	
0	0	1	0	1				x	x
0	0	1	1	0					
0	0	1	1	1					x
0	1	0	0	0		x			
0	1	0	0	1					x
0	1	0	1	0				x	
0	1	0	1	1				x	x
0	1	1	0	0		x	x	x	
0	1	1	0	1			x	x	x
0	1	1	1	0			x		
0	1	1	1	1			x		x
1	0	0	0	0	x	x	x		
1	0	0	0	1	x		x		x
1	0	0	1	0	x		x	x	
1	0	0	1	1	x		x	x	
1	0	1	0	0	x	x		x	
1	0	1	0	1	x			x	x
1	0	1	1	0	x				
1	0	1	1	1	x				
1	1	0	0	0		x			
1	1	0	0	1					x
1	1	0	1	0				x	
1	1	0	1	1				x	
1	1	1	0	0		x	x	x	
1	1	1	0	1			x	x	x
1	1	1	1	0			x		
1	1	1	1	1			x		

Note 7: Margaret Hamilton

Margaret Hamilton is an American computer scientist and systems engineer who led the MIT Software Engineering Division, which was responsible for developing the computer systems on board the Apollo 11 spacecraft. Hamilton and her team used a more robust version of punch cards known as core rope memory to programme these computers, which like all modern computers, stored information in binary arithmetic.

Note 8: William Stanley Jevons and the Logic Piano

William Stanley Jevons was an English economist and logician who is credited with the invention of the 'logic piano'; the first logic machine to solve complicated problems at an exceptional speed. Much like a musical piano, this device consisted of a series of black-and-white keys, which were used for entering premises. A face plate was placed above the keyboard and displayed the truth table. As the keys were struck, rods would mechanically remove the truth table entries inconsistent with the premises entered on the keys. The logical punch cards are a variant of the logical piano and were popularised by Martin Gardner in the mid-20th century.

Note 9: Premises for Activity 3

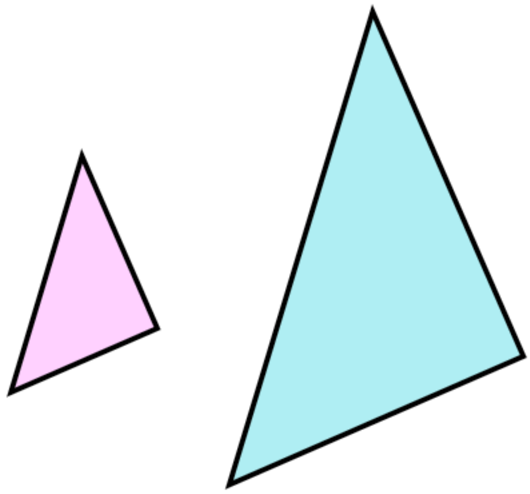
The problem below was obtained from George Boole's first publication on the theory of Boolean Algebra in 1854, which was entitled 'An Investigation of the Law of Thought'. In the book, Boole solves this problem using pen-and-paper (Boolean) algebra. However, the punch cards and truth tables use a similar principle.

Premise 1: Triangles whose corresponding angles are equal have their corresponding sides proportional, and vice versa.

The first premise states a property of triangles:

- Two triangles have the same angle at each of their corresponding vertices, *if and only if* ($\Leftrightarrow$) the corresponding sides of the two triangles are proportional.

Note: This statement does not mention anything about similarity.

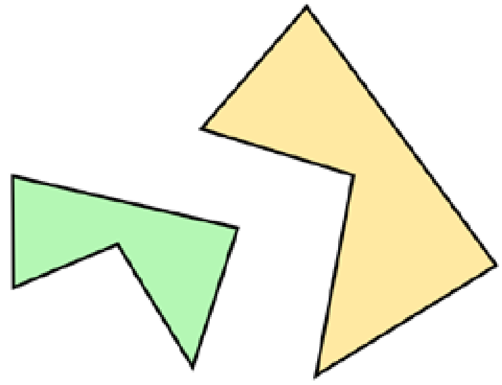


Premise 2: Similar figures consist of all whose corresponding angles are equal, and whose corresponding sides are proportional.

The second premise is a definition of what it means for two figures (polygons) to be similar:

- All the corresponding vertices have the same angle and
- All the corresponding sides are proportional.

Note: This statement does not mention anything about triangles.



You may wish to ask the students if the first statement still holds for shapes other than triangles (One counterexample would be a square and a rectangle, they have equal angles at every vertex (90°), but the sides are not proportional).

Using the following premises:

1. Triangles whose corresponding angles are equal have their corresponding sides proportional, and vice versa.
2. Similar figures consist of all whose corresponding angles are equal, and whose corresponding sides are proportional.

Which statement is true?

A. Triangles, for which their corresponding sides are proportional or corresponding angles are equal (but not both) are similar triangles.

B. Dissimilar triangles, for which their corresponding angles are equal, have corresponding sides that are disproportional.

C. Figures, for which their corresponding sides are proportional, and their corresponding angles are equal (but are not triangles) are similar.

D. Non-triangular, dissimilar figures could have corresponding sides proportional, and corresponding angles equal.

Note 10: Solutions for Activity 3

Note: This problem only involves four terms (S, T, Q and R), which gives a total of 16 possible combinations. You could, therefore, split a full deck of 32 punch cards into two smaller decks (one deck with 0 - 15 and another with 16 - 31) as this would allow for twice the number of student groups.

To represent these premises, we can let:

S	=	similar
T	=	triangles
Q	=	having corresponding angles equal
R	=	having corresponding sides proportional

Q1. From the second premise, which combinations should we eliminate?

Since the premise only makes a statement about triangles, we focus on cases with $\bar{T}$.

T	Q	R	Keep/discard
1	0	0	Keep
1	0	1	Discard
1	1	0	Discard
1	1	1	Keep

Any triangle that satisfies Q must also satisfy R and vice versa. Thus, we eliminate $T\bar{Q}R$ and $TQ\bar{R}$.

Q2. How could you remove the contradictory combinations from Q1 from the deck?

Labelling four slots with S, T, Q, and R. We first separate T with $\bar{T}$ (using the T slot). From the T pile, separate Q with $\bar{Q}$ (using the slot Q) to get two piles: TQ and $T\bar{Q}$. Further divide TQ and $T\bar{Q}$ (using the slot R) into four piles: TQR, $TQ\bar{R}$, $T\bar{Q}R$, and $T\bar{Q}\bar{R}$. Discard $TQ\bar{R}$ and $T\bar{Q}R$ then recombine the rest. You should be left with 12 cards.

Q3. From the second premise, which combinations should we eliminate?

S	Q	R	Keep/discard
0	0	0	Keep
0	0	1	Keep
0	1	0	Keep
0	1	1	Discard
1	0	0	Discard
1	0	1	Discard
1	1	0	Discard
1	1	1	Keep

We eliminate $\mathrm{S\bar{Q}\bar{R}}$, $\mathrm{S\bar{Q}R}$, and $\mathrm{SQ\bar{R}}$ because both conditions (Q and R) need to be satisfied for two figures to be similar. We also eliminate $\mathrm{\bar{S}QR}$ because if both conditions (Q and R) are satisfied, the two figures are similar by definition.

Q4. How could you remove the contradictory combinations from Q3 from the deck?

First, we separate S from $\mathrm{\bar{S}}$ (using the S slot). From the S pile, we would like to keep SQR. We first divide S into SQ and $\mathrm{S\bar{Q}}$ (using Q slot), then further divide SQ into SQR and $\mathrm{SQ\bar{R}}$ (using R slot). Keep the SQR pile and discard the rest.

Meanwhile, from the $\mathrm{\bar{S}}$ pile, we would like to eliminate $\mathrm{\bar{S}QR}$. We achieve this by separating the $\mathrm{\bar{S}}$ pile into $\mathrm{\bar{S}Q}$ and $\mathrm{\bar{S}\bar{Q}}$ (using slot Q), then further divide $\mathrm{\bar{S}Q}$ into $\mathrm{\bar{S}QR}$ and $\mathrm{\bar{S}Q\bar{R}}$ (using slot R). Discard $\mathrm{\bar{S}QR}$ and recombine $\mathrm{\bar{S}\bar{Q}}$, $\mathrm{\bar{S}Q\bar{R}}$ and SQR from above.

You should be left with 6 cards at this point. These are $\mathrm{\bar{S}\bar{T}\bar{Q}\bar{R}}$, $\mathrm{\bar{S}\bar{T}\bar{Q}R}$, $\mathrm{\bar{S}\bar{T}Q\bar{R}}$, $\mathrm{\bar{S}T\bar{Q}\bar{R}}$, $\mathrm{S\bar{T}QR}$, and STQR respectively. These are all the combinations that do not contradict the two premises we have.

Q5. Express each of the options (A, B, C, and D) of the question in terms of $\mathrm{S},\mathrm{\bar{S}},\mathrm{T},\mathrm{\bar{T}},\mathrm{Q},\mathrm{\bar{Q}},\mathrm{R},\mathrm{\bar{R}}$. Which statement is valid?

Each of the statements could be expressed as follows:

A. $\mathrm{T\bar{Q}R}$ and $\mathrm{TQ\bar{R}}$

B. $\mathrm{\bar{S}TQ\bar{R}}$

C. $\mathrm{S\bar{T}QR}$

D. $\mathrm{\bar{S}\bar{T}QR}$

The only valid statement is C ($S\bar{T}QR$) as it is one of the six cards that does not contradict the two premises.

A and B are invalid because any triangle that satisfies Q must also satisfy R and vice versa.

D is invalid because figures that satisfy Q and R, by definition, are similar.

Remark: Below is a table of cards that should be eliminated during each of the premises

S	T	Q	R	Premise 1	Premise 2
0	0	0	0		
0	0	0	1		
0	0	1	0		
0	0	1	1		X
0	1	0	0		
0	1	0	1	X	
0	1	1	0	X	
0	1	1	1		X
1	0	0	0		X
1	0	0	1		X
1	0	1	0		X
1	0	1	1		
1	1	0	0		X
1	1	0	1	X	X
1	1	1	0	X	X
1	1	1	1		

X represents cards that contradict the premise.

Sources and Additional Resources

Boole, G (1854). *An Investigation of the Law of Thought. Open Court.*

Gardner, M. (1969). Martin Gardner's new mathematical diversions from 'Scientific American'. London: Allen & Unwin.

http://www.upriss.org.uk/db/slides/logicpuzzle.pdf (Party invitation problem)

http://assets.cambridge.org/97805217/56075/excerpt/9780521756075_excerpt.pdf (Who is watching TV?)

http://history-computer.com/ModernComputer/thinkers/Jevons.html (Logic piano)

https://airandspace.si.edu/stories/editorial/rope-mother-margaret-hamilton (Margaret Hamilton)

Logic Machine: Activity 1

You have three friends John, Paul, and Mary, whom you would like to invite to your party.

However, there are a few premises to consider:

1. If you invite Mary, you must invite John.
2. If you don't invite Paul, you must invite both John and Mary.
3. You must invite either John or Mary, but not both.

The Party Invitation

Q1. Fill in the last two columns of the truth table below by writing ' ✕ ' to indicate combination that contradicts the premise and ' ✔ ' to indicate combination that does not contradict the premise in the respective column.

John	Paul	Mary	Premise 1	Premise 2	Premise 3
0	0	0	✔		
0	0	1	✕		
0	1	0	✔		
0	1	1	✕		
1	0	0	✔		
1	0	1	✔		
1	1	0	✔		
1	1	1	✔		

1 = invite, 0 = does not invite

Q2. Based on the table above, who should you invite to the party?

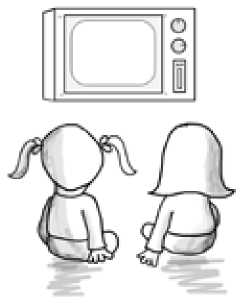
Logic Machine: Activity 2

Albert and his wife Breda have three children: Ciara, David and Ellie.

1. If Albert is watching television, so is his wife.
2. Either David or Ellie, or both of them, are watching television.
3. Either Breda or Ciara, but not both, is watching television.
4. David and Ciara are either both watching television or both not watching television.
5. If Ellie is watching television, then Albert and David are also watching

Who is watching TV?

Q1. How many possible scenarios (combinations of who is watching and who is not watching television) could we have for this TV watching session? How many cards do you need?



Q2. From the first premise, which combination should we eliminate?
(Hint: does the statement say anything about Albert if Breda is watching television?)

AB – Albert and Breda both watching TV.

$\mathrm{A\bar{B}}$ – Albert is watching TV and Breda is not watching TV.

$\mathrm{\bar{A}B}$ – Breda is watching TV and Albert is not watching TV.

$\mathrm{\bar{A}\bar{B}}$ – Albert and Breda both not watching TV.

Q3. How could you remove the contradictory combination from Q2 from the deck? You may wish to discuss this with your group.

Q4. From the second premise, what combinations should we eliminate? (Write ' ✕ ' or ' ✓ ' in the last column to indicate discard or keep respectively).

David is watching TV	Ellie is watching TV	Keep/discard
0	0	
0	1	
1	0	
1	1	

Q5. How could you remove the contradictory combination from Q4 from the deck? You may wish to discuss this with your group.

Q6. Based on the third premise, what combinations should we eliminate? (Hint: There are multiple combinations that should be eliminated).

Breda is watching TV	Ciara is watching TV	Keep/discard
0	0	
0	1	
1	0	
1	1	

Q7. How could you remove the contradictory combinations from Q6 from the deck? You may wish to discuss this with your group.

Q8. Based on the fourth premise, what combinations should we eliminate?

Ciara is watching TV	David is watching TV	Keep/discard
0	0	
0	1	
1	0	
1	1	

Q9. How could you remove the contradictory combinations from Q8 from the deck? You may wish to discuss this with your group.

Q10. Based on the final premise, what combinations should we eliminate? (Beware that this premise involves 3 people).

Ellie is watching TV	Albert is watching TV	David is watching TV	Keep/discard
1	0	0	
1	0	1	
1	1	0	
1	1	1	

Q11. How could you remove the contradictory combinations from Q10 from the deck? You may wish to discuss this with your group.

Q12. Can you now determine who is watching tv?

Logic Machine: Activity 3

Using the following premises:

1. Triangles whose corresponding angles are equal have their corresponding sides proportional, and vice versa.
2. Similar figures consist of all whose corresponding angles are equal, and whose corresponding sides are proportional.

Which statement is true?

A. Triangles, for which their corresponding sides are proportional or corresponding angles are equal (but not both) are similar triangles.

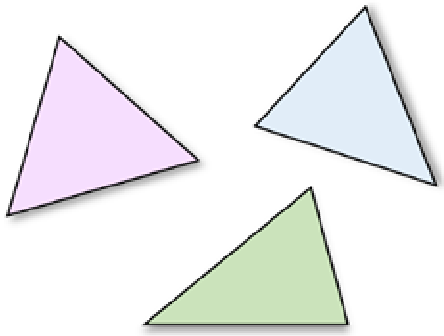
B. Dissimilar triangles, for which their corresponding angles are equal, have corresponding sides that are disproportional.

C. Figures, for which their corresponding sides are proportional, and their corresponding angles are equal (but are not triangles) are similar.

D. Non-triangular, dissimilar figures could have corresponding sides proportional, and corresponding angles equal.

To represent these premises, we can let

S	=	similar
T	=	triangles
Q	=	having corresponding angles equal
R	=	having corresponding sides proportional



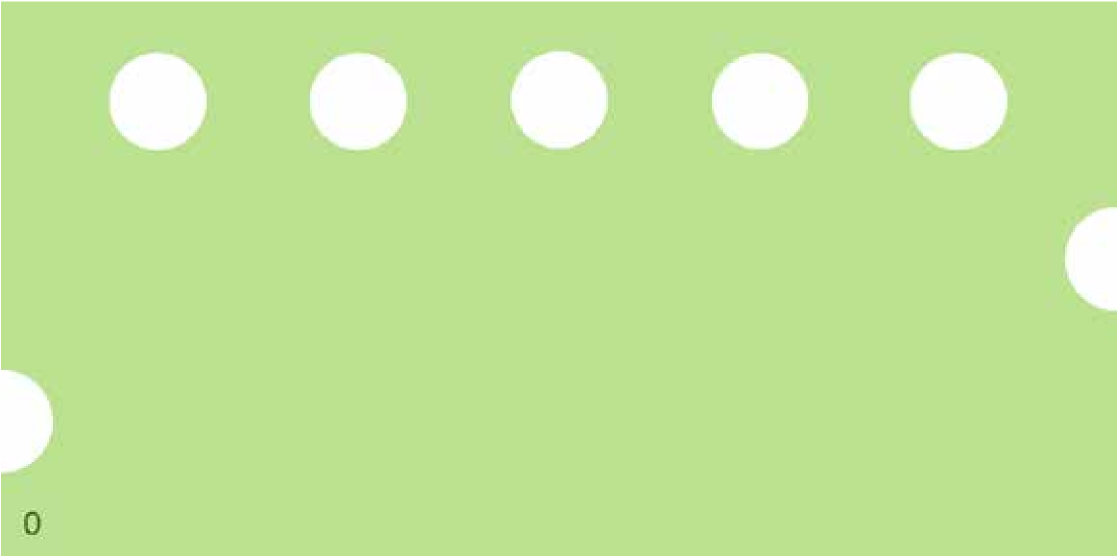
Q1. From the second premise, which combinations should we eliminate? (there is more than one combination to be eliminated)

Q2. How could you remove the contradictory combinations from Q1 from the deck?

Q3. From the second premise, which combinations should we eliminate? (there is more than one combination to be eliminated).

Q4. How could you remove the contradictory combinations from Q3 from the deck?

Q5. Express each of the options (A, B, C, and D) of the question in terms of $S, \bar{S}, T, \bar{T}, Q, \bar{Q}, R, \bar{R}$. Which statement is valid?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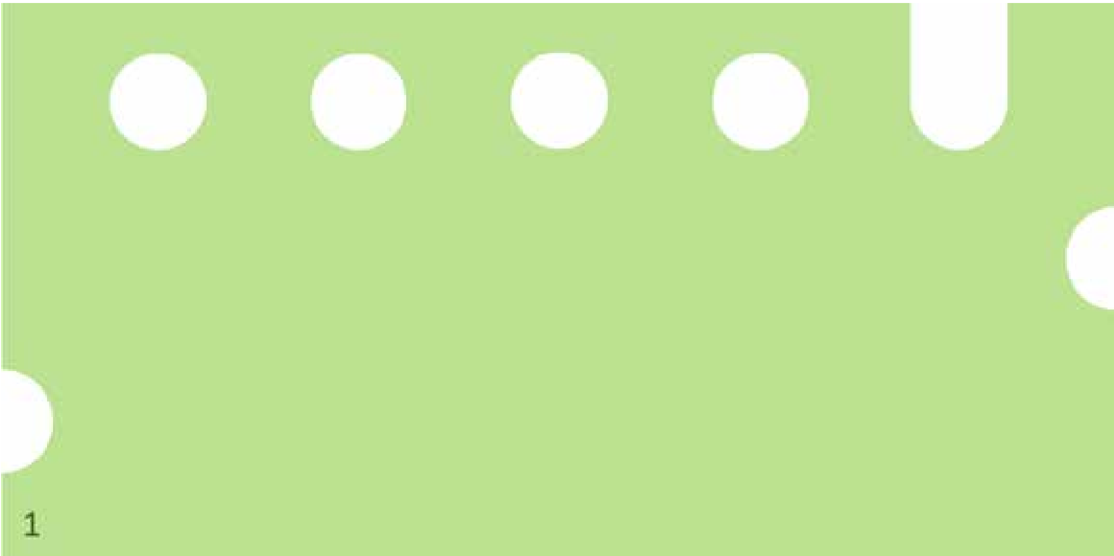
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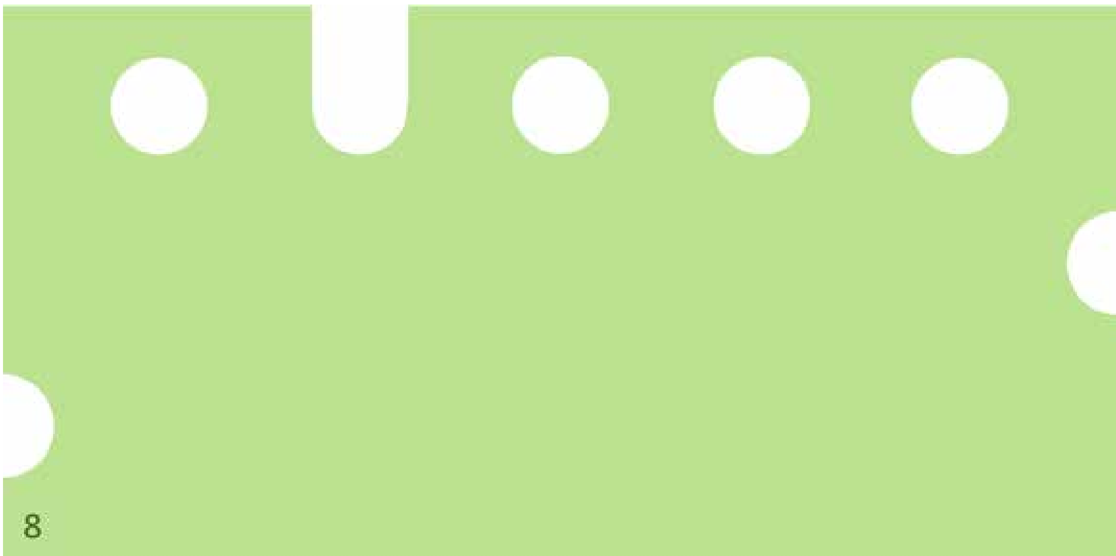
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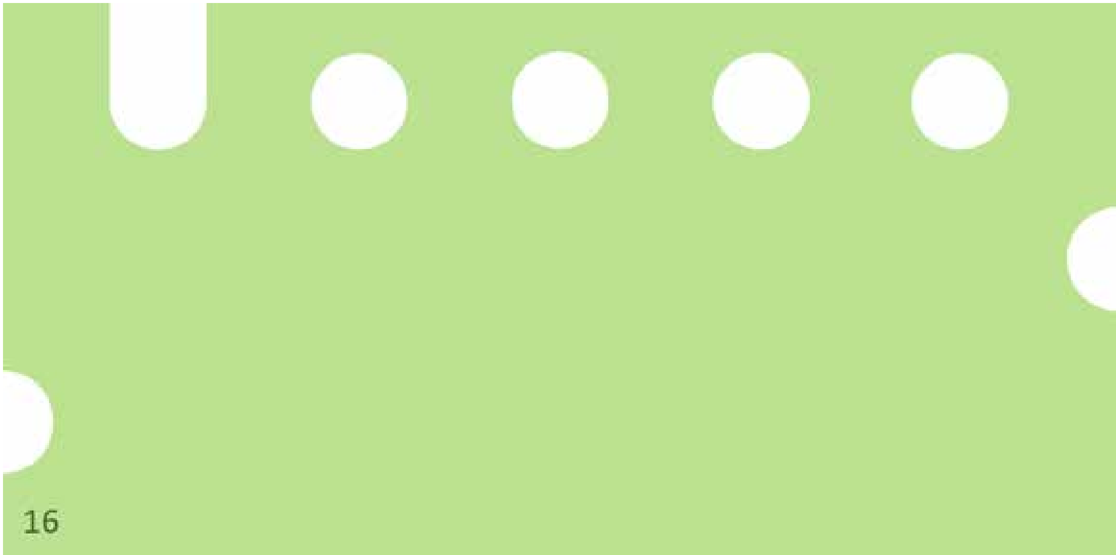
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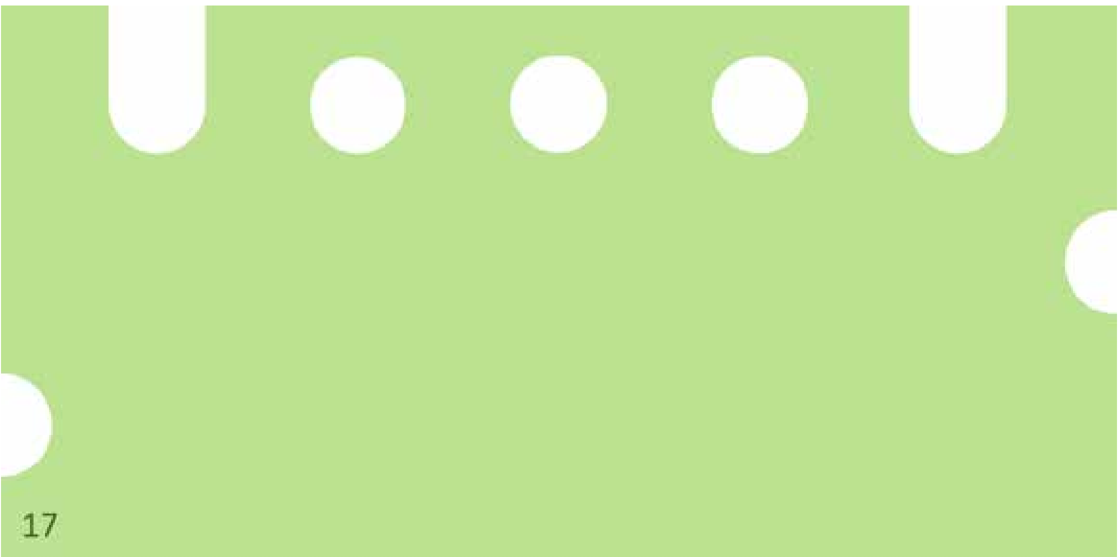
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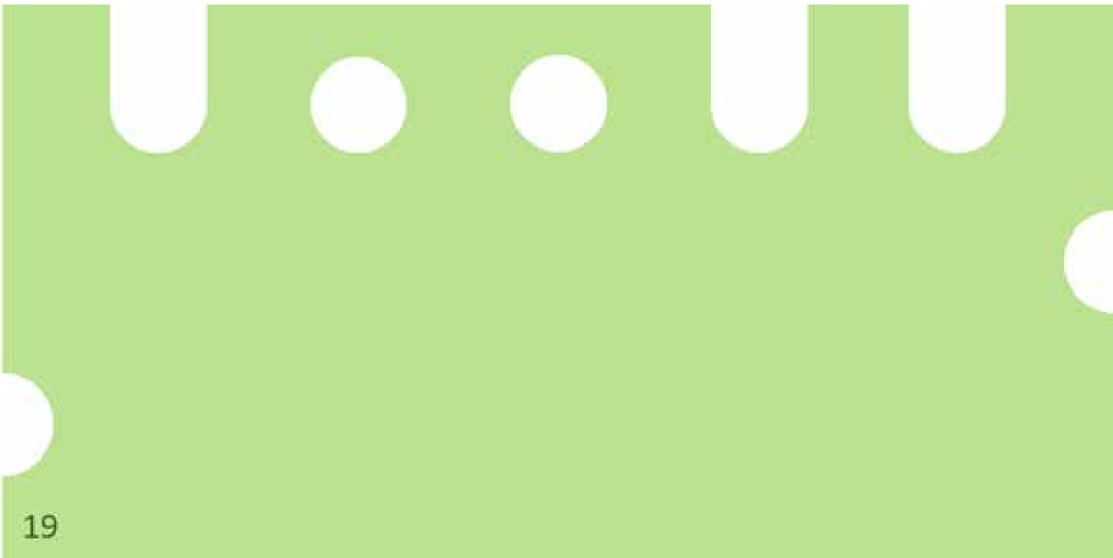
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UCD Undergraduate Student Volunteers for Maths Sparks 2018

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Christine Coffee	Stage 2 BSc. College of Science
Cian Jameson	Stage 4 BSc. School of Mathematics & Statistics, College of Science
Claire Mullen	Stage 3 BSc. School of Mathematics & Statistics, College of Science
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Jasmine Chang Yi Hua	Stage 3 BSc. School of Mathematics & Statistics, College of Science
Khang Ee Pang	Stage 3 BSc. School of Mathematics & Statistics, College of Science
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Martha Kennedy-Ralph	Stage 2 BA. College of Social Sciences & Law
Thomas Creavin	Stage 2 School of Computer Science
Kerry Brooks	Stage 4 BSc. School of Mathematics & Statistics, College of Science
Siobhan Carty	Stage 4 BSc. School of Mathematics & Statistics, College of Science

Participating Schools

Alexandra College, Milltown
Cabinteely Community College, Cabinteely
Caritas College, Ballyfermot
Greenhills College, Walkinstown
Killinarden Community School, Tallaght
Kylemore College, Ballyfermot
Mount Seskin Community College, Tallaght
St. Aidan's Community School, Tallaght
St. Dominic's Secondary School, Ballyfermot
St. John's College, Ballyfermot
St. Mark's Community School, Tallaght
St. Paul's Secondary School, Greenhills
St. Tiernan's Community School, Balally
Tallaght Community School
Temple Carrig School, Greystones

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Dr Aoibhinn Ní Shúilleabháin

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