Inequalities

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Basic Principle of Inequalities: For any real number x, we have

$$x^2 \geq 0 \text{ , with equality if and only if } x = 0 \text{ .}$$

Example. Prove that $x^2 + y^2 + z^2 \geq xy + yz + zx$, and determine when equality occurs.

Solution. Note that

$$(x - y)^2 \geq 0$$

$$(y - z)^2 \geq 0$$

$$(z - x)^2 \geq 0 \text{ .}$$

Adding these three inequalities yields

$$2(x^2 + y^2 + z^2) - 2(xy + yz + zx) \geq 0 \text{ .} \tag{1}$$

Equality will occur in (1) if and only if equality occurs in all three of the inequalities that were added. Therefore, equality occurs if and only if $x = y = z$.

Example. For any two positive real numbers x and y, we have $(x-y)^2 \geq 0$, and so $x^2+y^2-2xy \geq 0$. Writing this as $x^2+y^2+2xy \geq 4xy$, we get $\left(\frac{x+y}{2}\right)^2 \geq xy$. Taking the square root of both sides yields

$$\frac{x+y}{2} \geq \sqrt{xy}\ .$$

where by convention, $\sqrt{\cdot}$ denotes the *positive* square root. This inequality has a special name.

The Arithmetic Mean – Geometric Mean (AM-GM) Inequality:

For any two positive real numbers x and y, we have

$$\frac{x+y}{2} \geq \sqrt{xy}$$

with equality if and only if $x = y$.

The quantity of the LHS is called the *arithmetic mean* of the two numbers x and y. The quantity of the LHS is called the *geometric mean* of the two numbers x and y. They can be regarded as providing two different ways of "averaging" a pair of numbers.

This gives a new "Law of Averages": *Some averages are bigger than others!*

Remark: This result has the following interpretations:

- The *minimum* value of the *sum* of two positive quantities whose *product* is fixed occurs when both are equal.
- The *maximum* value of the *product* of two positive quantities whose *sum* is fixed occurs when both are equal.
- A geometric interpretation of this result is that "the rectangle of largest area, with a fixed perimeter, is a square".

Example. Find the minimum of

$$\frac{x}{y} + \frac{y}{x}$$

where x and y are positive.

Solution. By the AM-GM inequality,

$$\begin{aligned} \frac{x}{y} + \frac{y}{x} &\geq 2\sqrt{\frac{x}{y} \cdot \frac{y}{x}} \\ &= 2 \, . \end{aligned}$$

The minimum occurs when $\frac{x}{y} = \frac{y}{x}$, i.e., when $x^2 = y^2$. Since x and y are both positive, this occurs if and only if $x = y$.

The Arithmetic Mean – Geometric Mean (AM-GM) Inequality (more than two variables):

Suppose we have n positive real numbers $x_1, x_2, \ldots, x_n$. Then

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq (x_1 x_2 \cdots x_n)^{\frac{1}{n}}$$

with equality if and only if all of the numbers $x_1, x_2, \ldots, x_n$ are equal.

Remark: This result has the following interpretations:

- The *minimum* value of the *sum* of positive quantities whose *product* is fixed occurs when all are equal.
- The *maximum* value of the *product* of positive quantities whose *sum* is fixed occurs when all are equal.

Example. Find the minimum of

$$\frac{50}{x} + \frac{20}{y} + xy$$

where $x, y > 0$.

Solution. By AM-GM,

$$\begin{aligned} \frac{50}{x} + \frac{20}{y} + xy &\geq 3\sqrt[3]{\frac{50}{x} \cdot \frac{20}{y} \cdot xy} \\ &= 3\sqrt[3]{1000} \\ &= 30\ . \end{aligned}$$

The minimum occurs when $\frac{50}{x} = \frac{20}{y} = xy$, i.e., when $x = 5$ and $y = 2$.

Example.

Maximize $xy\,(72-3x-4y)$, where $x, y > 0$ and $3x+4y < 72$.

Solution. We seek to maximize the product of three positive quantities. Note that the sum of the three quantities is equal to

$$x + y + (72 - 3x - 4y) = 72 - 2x - 3y\ .$$

This is NOT a constant! However, we can rearrange the product as

$$\frac{1}{12}\,(3x)\,(4y)\,(72 - 3x - 4y)$$

Thus by AM-GM, the maximum occurs when $3x = 4y = 72 - 3x - 4y$, i.e., when $3x = 72 - 6x$. This yields $9x = 72$, or $x = 8$. Thus $y = 6$ and the maximum value is $\frac{1}{12} \cdot (24)^3 = 1152$.

Example.

Find the maximum value of $f(x) = (1-x)(1+x)^2$, where $0 \leq x \leq 1$.

Solution. Writing $f(x) = (1-x)(1+x)(1+x)$, we seek to maximize the product of three positive quantities. Note that the sum of the three quantities is equal to

$$(1-x)+(1+x)+(1+x) = 3+x \ .$$

This is NOT a constant! However, we can rearrange the product as

$$f(x) = \frac{1}{2}(2-2x)(1+x)(1+x) \ .$$

We then have, by AM-GM,

$$\sqrt[3]{(2-2x)(1+x)(1+x)} \leq \frac{(2-2x)+(1+x)+(1+x)}{3} = \frac{4}{3}$$

and so

$$(2-2x)(1+x)(1+x) \leq \left(\frac{4}{3}\right)^3 = \frac{64}{27}$$

and so $f(x) \leq \frac{32}{27}$. The maximum is achieved when $2-2x = 1+x = 1+x$, i.e., when $x = 1/3$.

Two More “Averages”:

The **Harmonic Mean** of n numbers $x_1, x_2, \ldots, x_n$ is given by

$$\mathrm{HM} = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}$$

and their **Root-Mean-Square** is given by

$$\mathrm{RMS} = \sqrt{\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}}\ .$$

If all the numbers $x_1, x_2, \ldots, x_n$ are positive, then we have

$$\min\{x_1, \ldots, x_n\} \leq \mathrm{HM} \leq \mathrm{GM} \leq \mathrm{AM} \leq \mathrm{RMS} \leq \max\{x_1, \ldots, x_n\}$$

with equality in each case if and only if all of the numbers $x_1, x_2, \ldots, x_n$ are equal.

Special case: for two positive numbers x and y

$$\min\{x, y\} \leq \frac{2xy}{x+y} \leq \sqrt{xy} \leq \frac{x+y}{2} \leq \sqrt{\frac{x^2+y^2}{2}} \leq \max\{x, y\}\ .$$

Exercise: Prove the above special case (*all* inequalities)!

Next we will prove a famous elementary inequality called **The Rearrangement Inequality**. We will then show that this inequality has some far-reaching consequences!

Motivating Example (Part 1). Banknotes are available in the denominations of EUR5 and EUR10. You are allowed to take 3 banknotes of one type, and 7 banknotes of the other type. How should you choose in order to maximize the amount of money you have?

Answer. Choose 3 EUR5 notes, and 7 EUR10 notes. "Obvious"!

Justification. Because

$$3 \cdot 5 + 7 \cdot 10 > 3 \cdot 10 + 7 \cdot 5 \ .$$

This example motivates the following result.

The Rearrangement Inequality (Case of two variables): Let $a < b$ and $x < y$. Then

$$ax + by > ay + bx \ .$$

Proof: Note that $b - a > 0$ and also $y - x > 0$. Therefore

$$(b-a)(y-x) > 0 \ .$$

Expanding this product yields

$$ax + by - ay - bx > 0 \ ,$$

giving the result.

Motivating Example for the General Case. Banknotes are available in the denominations of EUR5, EUR10 and EUR20. You are allowed to take 3 banknotes of one type, 7 banknotes of a second type, and 9 banknotes of the third type. How should you choose in order to maximize the amount of money you have?

Answer. Choose 3 EUR5 notes, 7 EUR10 notes, and 9 EUR20 notes. Again, "obvious"!

Justification. Because

$$3 \cdot 5 + 7 \cdot 10 + 9 \cdot 20 > 3 \cdot x + 7 \cdot y + 9 \cdot z \ ,$$

where x, y, z is any rearrangement of $5, 10, 20$.

This example motivates the following general result.

The Rearrangement Inequality:

Suppose that

- The n numbers $a_1, a_2, \ldots, a_n$ are in *increasing order*, i.e., $a_1 < a_2 < \cdots < a_n$
- The n numbers $b_1, b_2, \ldots, b_n$ are also in *increasing order*, i.e., $b_1 < b_2 < \cdots < b_n$

If $x_1, x_2, \ldots, x_n$ is a rearrangement (or permutation) of the numbers $b_1, b_2, \ldots, b_n$, then

$$(2) \qquad a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq a_1b_1 + a_2b_2 + \cdots + a_nb_n$$

with equality if and only if the numbers $x_1, x_2, \ldots, x_n$ are in *increasing order*, i.e., if and only if $x_1 = b_1$, $x_1 = b_1$, ..., $x_n = b_n$.

In other words, the maximum of the *mixed sum*

$$M = a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

is equal to the *forward-ordered sum*

$$F = a_1b_1 + a_2b_2 + \cdots + a_nb_n \ .$$

Proof. Suppose we consider any mixed sum

$$M = a_1x_1 + a_2x_2 + \cdots + a_nx_n \ .$$

Suppose that the arrangement $x_1, x_2, \ldots, x_n$ maximizes the mixed sum. Suppose also that we can find two numbers x_i and x_j such that $a_i < a_j$ but $x_i > x_j$. Suppose we swap x_i with x_j. What happens to the mixed sum?

The mixed sum beforehand equals

$$M = a_1x_1 + a_2x_2 + \cdots + a_ix_i + \cdots + a_jx_j + \cdots + a_nx_n$$

and after the swap equals

$$M' = a_1x_1 + a_2x_2 + \cdots + a_ix_j + \cdots + a_jx_i + \cdots + a_nx_n$$

Does the mixed sum increase? In other words, is $M' > M$? Well, this will be true if

$$a_ix_j + a_jx_i > a_ix_i + a_jx_j \ .$$

But this must be true since

$$(a_j - a_i)(x_i - x_j) > 0 \ .$$

But then the mixed sum after the swap is larger than before the swap. This contradicts our initial assumption that "we can find two numbers x_i and x_j such that $a_i < a_j$ but $x_i > x_j$". If this assumption does not hold, then we must have $x_i < x_j$ whenever $a_i < a_j$.

This shows that the unique arrangement which maximizes the mixed sum is $x_1 = b_1$, $x_2 = b_2$, ..., $x_n = b_n$, i.e., when the numbers $x_1, x_2, \ldots, x_n$ are in *increasing order*. This completes the proof.

Example. [From earlier] Prove that $x^2 + y^2 + z^2 \geq xy + yz + zx$, and determine when equality occurs.

Solution. Note that (x, y, z) and (x, y, z) are in the same order, but (x, y, z) and (y, z, x) are not. Thus, applying the Rearrangement Inequality, we obtain

$$x \cdot x + y \cdot y + z \cdot z \geq x \cdot y + y \cdot z + z \cdot x$$

and the result is proved.

Example. ["Nesbitt's Inequality"] Prove that for positive numbers a, b, c,

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2} \ .$$

Solution. Let $s = a + b + c$. Then

$$\begin{aligned} X &= \frac{a}{s-a} + \frac{b}{s-b} + \frac{c}{s-c} \\ &= a \cdot \frac{1}{s-a} + b \cdot \frac{1}{s-b} + c \cdot \frac{1}{s-c} \ . \end{aligned}$$

Note that (a, b, c) and $\left(\frac{1}{s-a}, \frac{1}{s-b}, \frac{1}{s-c}\right)$ are in the same order (Exercise: prove this!). Therefore,

$$X = a\cdot\frac{1}{s-a}+b\cdot\frac{1}{s-b}+c\cdot\frac{1}{s-c} \geq x\cdot\frac{1}{s-a}+y\cdot\frac{1}{s-b}+z\cdot\frac{1}{s-c}$$

where (x, y, z) is any rearrangement of (a, b, c). So

$$X \geq \frac{b}{s-a} + \frac{c}{s-b} + \frac{a}{s-c}$$

and

$$X \geq \frac{c}{s-a} + \frac{a}{s-b} + \frac{b}{s-c}$$

Adding the two previous equations yields

$$2X \geq \frac{b+c}{s-a} + \frac{a+c}{s-b} + \frac{a+b}{s-c} = 3$$

i.e.

$$X \geq 3/2 \ .$$

Motivating Example for a Related Result. Banknotes are available in the denominations of EUR5, EUR10 and EUR20. You are allowed to take 3 banknotes of one type, 7 banknotes of a second type, and 9 banknotes of the third type. How should you choose in order to **minimize** the amount of money you have?

Answer. Choose 9 EUR5 notes, 7 EUR10 notes, and 3 EUR20 notes.

Corollary to the Rearrangement Inequality:

Suppose that

- The n numbers $a_1, a_2, \ldots, a_n$ are in *increasing order*, i.e., $a_1 < a_2 < \cdots < a_n$
- The n numbers $b_1, b_2, \ldots, b_n$ are also in *increasing order*, i.e., $b_1 < b_2 < \cdots < b_n$

If $x_1, x_2, \ldots, x_n$ is a rearrangement (or permutation) of the numbers $b_1, b_2, \ldots, b_n$, then

$$(3) \qquad a_1x_1 + a_2x_2 + \cdots + a_nx_n \geq a_1b_n + a_2b_{n-1} + \cdots + a_nb_1$$

with equality if and only if $x_1 = b_n$, $x_1 = b_{n-1}$, …, $x_n = b_1$.

This tells us that the minimum of the *mixed sum*

$$M = a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

is equal to the *reverse-ordered sum*

$$R = a_1b_n + a_2b_{n-1} + \cdots + a_nb_1 \ .$$

Proof of the Corollary to the Rearrangement Inequality:

Applying the Rearrangement Inequality (2) with $-b_n \leq -b_{n-1} \leq$ $-b_1$ in place of $b_1 \leq b_2 \leq \cdots \leq b_n$ we obtain

(4)

$$a_1(-x_1)+a_2(-x_2)+\cdots+a_n(-x_n) \leq a_1(-b_n)+a_2(-b_{n-1})+\cdots+a_n(-b_1)$$

Here we note that if $x_1, x_2, \ldots, x_n$ is a rearrangement of the numbers $b_1, b_2, \ldots, b_n$, then $-x_1, -x_2, \ldots, -x_n$ is a rearrangement of the numbers $-b_1, -b_2, \ldots, -b_n$.

Simplifying (4) leads to the desired result.

Exercises.

(1) Show that $4x^2+1 \geq 4x$ and determine when equality occurs.

(2) If a and b are positive real numbers, prove that

$$(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \geq 4$$

and determine when equality occurs.

(3) Find the minimum value of

$$x+\frac{8}{y(x-y)}$$

where $y>0$ and $x>y$.

(4) Show that if a and b are positive real numbers, then $(a+2b)(2a+b)>8ab$.

(5) Find the positive number whose square exceeds its cube by the greatest amount.

(6) Prove that for positive real numbers x, y, z,

$$\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x} \leq \frac{1}{2}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) .$$

(7) Prove that for positive real numbers a, b, c,

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$$

and determine when equality occurs.

(8) Let a, b, c be positive real numbers whose sum is 1. Prove that

$$\frac{a}{(a+1)(b+1)} + \frac{b}{(b+1)(c+1)} + \frac{c}{(c+1)(a+1)} \geq \frac{3}{4}\,.$$

For further reading, click here:

Wikipedia entry on AM-GM

Wikipedia entry on the Rearrangement Inequality