THE PRINCIPLE OF INDUCTION

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The Principle of Induction: Let a be an integer, and let $P(n)$ be a statement (or proposition) about n for each integer $n \geq a$. The principle of induction is a way of proving that $P(n)$ is true for all integers $n \geq a$. It works in two steps:

(a) [**Base case:**] Prove that $P(a)$ is true.
(b) [**Inductive step:**] Assume that $P(k)$ is true for some integer $k \geq a$, and use this to prove that $P(k+1)$ is true.

Then we may conclude that $P(n)$ is true for all integers $n \geq a$.

This principle is very useful in problem solving, especially when we observe a *pattern* and want to prove it.

The trick to using the Principle of Induction properly is to spot *how to use* $P(k)$ to prove $P(k+1)$. Sometimes this must be done rather ingeniously!

Problem 1. Prove that for any integer $n \geq 1$,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \ .$$

Solution. Let $P(n)$ denote the proposition to be proved. First let's examine $P(1)$: this states that

$$1 = \frac{1(2)}{2} = 1$$

which is correct.

Next, we assume that $P(k)$ is true for some positive integer k, i.e.

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2} \ .$$

and we want to use this to prove $P(k+1)$, i.e.

$$1 + 2 + 3 + \cdots + (k+1) = \frac{(k+1)(k+2)}{2} \ .$$

Taking the LHS and using $P(k)$,

$$\begin{aligned}
1 + 2 + 3 + \cdots + (k+1) &= (1 + 2 + 3 + \cdots + k) + (k+1) \\
&= \frac{k(k+1)}{2} + (k+1) \\
&= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2}
\end{aligned}$$

and thus $P(k+1)$ is true. This completes the proof.

Problem 2. Find a formula for the sum of the first n odd numbers.

Solution. Note that this time we are not told the formula that we have to prove; we have to find it ourselves! Let's try some small numbers and see if a pattern emerges:

$$1 = 1; \quad 1 + 3 = 4; \quad 1 + 3 + 5 = 9;$$

$$1 + 3 + 5 + 7 = 16; \quad 1 + 3 + 5 + 7 + 9 = 25;$$

We conjecture (guess) that the sum of the first n odd numbers is equal to n^2. Now let's prove this proposition using the principle of induction; call it $P(n)$.

Our statement $P(n)$ is that

$$1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2 \,.$$

First we prove the base case $P(1)$, i.e.

$$1 = 1^2$$

This is certainly true. Now we assume that $P(k)$ is true, i.e.

$$1 + 3 + 5 + 7 + \cdots + (2k - 1) = k^2 \,.$$

and consider $P(k + 1)$:

$$1 + 3 + 5 + 7 + \cdots + (2k + 1) = (k + 1)^2 \,.$$

Taking the LHS and using $P(k)$,

$$\begin{aligned} 1+3+5+\cdots+(2k+1) &= (1+3+5+\cdots+(2k-1))+(2k+1) \\ &= k^2+(2k+1) \\ &= (k+1)^2 . \end{aligned}$$

and thus $P(k+1)$ is true. This completes the proof.

Exercise 1. Show that for all $n \geq 1$,

$$1^2+3^2+5^2+\cdots+(2n-1)^2=\frac{n(4n^2-1)}{3} .$$

Problem 3. For any positive integer n, find the largest power of 2 that divides $(n+1)(n+2)\cdots(2n)$.

Solution. Let $f(n) = (n+1)(n+2)\cdots(2n)$. First, let's find the answer for $n = 1, 2, 3, 4$ to see if any pattern emerges:

$$n = 1: \quad f(1) = 2 \text{ is divisible by } 2^1$$

$$n = 2: \quad f(2) = 3 \cdot 4 \text{ is divisible by } 2^2$$

$$n = 3: \quad f(3) = 4 \cdot 5 \cdot 6 \text{ is divisible by } 2^3$$

$$n = 4: \quad f(4) = 5 \cdot 6 \cdot 7 \cdot 8 \text{ is divisible by } 2^4$$

So it seems that the largest power of 2 dividing $f(n)$ is 2^n. Now, let's prove this by induction.

The base case $n = 1$ is already done above. Assume that the result holds for $n = k$, i.e., that the largest power of 2 dividing $f(k) = (k+1)(k+2)\cdots(2k)$ is 2^k for some $k \geq 1$. Now look at

$$\begin{aligned} f(k+1) &= (k+2)(k+3)\cdots(2k)(2k+1)(2k+2) \\ &= [(k+1)(k+2)\cdots(2k)] \cdot \left[\frac{(2k+1)(2k+2)}{k+1}\right] \\ &= 2(2k+1)f(k) \end{aligned}$$

Since $2k+1$ is odd, and the highest power of 2 dividing $f(k)$ is 2^k, it follows that the highest power of 2 dividing $f(k+1)$ is 2^{k+1}. This completes the proof.

Alternative Solution. Note that

$$\begin{aligned} \frac{f(n)}{2^n} &= \frac{(n+1)(n+2)\cdots(2n)}{2^n} \\ &= \frac{1\cdot 2\cdot 3\cdots 2n}{2^n\cdot 1\cdot 2\cdot 3\cdots n} \\ &= \frac{1\cdot 2\cdot 3\cdots 2n}{2\cdot 4\cdot 6\cdots 2n} \\ &= 1\cdot 3\cdot 5\cdot 7\cdots(2n-1)\ . \end{aligned}$$

This is the product of all the *odd* integers from 1 to $2n-1$.

Exercise 2. Show that for all $n \geq 1$, we have $f(n) = g(n)$, where

$$f(n) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}$$

and

$$g(n) = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n}\ .$$

Exercise 3.

Prove that every number in the sequence

$$1007, 10017, 100117, 1001117, 10011117, \ldots$$

is divisible by 53.

Problem 4.

(a) Consider a circle and l lines in the plane, such that every line intersects the circle at two distinct points. Let p denote the number of points *strictly inside* the circle at which a pair of lines intersect each other. Also, let r denote the number of regions into which the circle is divided by the lines. Prove that

$$r = l + p + 1 \ .$$

Solution.

(a) We will first prove that $r = l + p + 1$ by induction on the number of lines.

The base case $l = 0$ is trivial; with no lines, there are no points of intersection inside the circle ($p = 0$) and the number of regions is $r = 1$ (the circle itself).

Suppose the relationship $r = l + p + 1$ is valid for some number l of lines. We will show that it remains valid if another line is added.

Let's add a new line to the picture. Suppose that it intersects the other lines at s points *inside* the circle (note that s could be zero). These s points of intersection split the new line into $s+1$ segments within the circle, and each segment splits an old region into two new regions.

Thus l increases by 1, p increases by s, and r increases by $s+1$. The formula $r = l + p + 1$ remains valid since both sides increase by $s+1$.

Therefore, by the principle of induction, the result holds for any number of lines $l \geq 1$.

(b) Let n be a positive integer. Place n points on the circumference of a circle, and draw all possible chords through pairs of these points. Assume that no three chords are concurrent (meet in a single point). Let a_n be the number of regions into which the circle is divided. Find a formula for a_n.

Solution.

We will use the result from part (a). Here we have:

- By definition, the number of regions is $r = a_n$;
- Each pair of points on the circle determines a unique line, so $l = \binom{n}{2}$;
- Each set of 4 points on the circle produces a unique intersection point inside the circle, so $p = \binom{n}{4}$.

Thus we obtain

$$a_n = \binom{n}{2} + \binom{n}{4} + 1 \ .$$

Problem 5. Every road in Uniland is one-way. Every pair of cities is connected by exactly one direct road. Show that there exists a city which can be reached from every other city either directly of via at most one other city.

Solution. Suppose there are n cities in Uniland. We will prove this by induction on n, starting with the base case $n = 2$. So first we prove that the proposition holds for 2 cities. This is easy since if there are only two cities A and B with a road from A to B, then B satisfies the conditions of the problem.

A city satisfying the conditions of the problem will be called a H-city. Next we assume that the result holds for k cities. This means that among the k cities there must be a H-city; let's call it A.

This means that every other city in Uniland has a road going directly to A (in which case we call it a D-city for A), or else a route going to A using some D-city X (in which case we call it an N-city for A). So every city in Uniland is either a D-city or an N-city for A.

Next we add one more city to Uniland, call this city P. We use the following reasoning:

Case 1: If a road goes from P to A, then P is a D-city for A. Therefore A is a H-city for the new problem.

Case 2: Let X be a D-city. If there is a road from P to D, then P is an N-city for A. Therefore A is a H-city for the new problem.

Case 3: The only other possibility is that roads go from A to P and from every D-city of A to P. But there is also a direct road from every N-city of A to some D-city of A. And so P is a H-city for the new problem.

Alternative Solution using the "Extremal Principle".

Let m be the *maximum* number of direct roads leading into any city, and let M be a city for which this maximum is attained.

Let D be the set of m cities with direct connections into M.

Let R be the set of all cities apart from M and the cities in D.

If R is empty, then M is the required city. If $X \in R$, then there is a city E in D such that a road leads from E in D, so that is it possible to reach M from X via D.

If such a city E did not exist, this would mean that all cities in D connect directly to X.

Since M also connects directly to X, there are $m+1$ direct roads into X. This is a contradiction, since the maximum number of direct roads leading into any city is m.

Therefore, *every city with the maximum number of entering roads* satisfies the conditions of the problem.

Exercise 4 (BMO Round 1, 1997).

For positive integers n, the sequence $a_1, a_2, a_3, \ldots$ is defined by $a_1 = 1$ and

$$a_n = \left(\frac{n+1}{n-1}\right)(a_1 + a_2 + \cdots + a_{n-1})$$

for $n > 1$. Determine the value of a_{1997}.