THE PRINCIPLE OF INDUCTION

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The Principle of Induction: Let a be an integer, and let $P(n)$ be a statement (or proposition) about n for each integer $n \geq a$. The principle of induction is a way of proving that $P(n)$ is true for all integers $n \geq a$. It works in two steps:

(a) [**Base case:**] Prove that $P(a)$ is true.
(b) [**Inductive step:**] Assume that $P(k)$ is true for some integer $k \geq a$, and use this to prove that $P(k+1)$ is true.

Then we may conclude that $P(n)$ is true for all integers $n \geq a$.

This principle is very useful in problem solving, especially when we observe a *pattern* and want to prove it.

The trick to using the Principle of Induction properly is to spot *how to use* $P(k)$ to prove $P(k+1)$. Sometimes this must be done rather ingeniously!

Problem 1. Prove that for any integer $n \geq 1$,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \; .$$

Solution. Let $P(n)$ denote the proposition to be proved. First let's examine $P(1)$: this states that

$$1 = \frac{1(2)}{2} = 1$$

which is correct.

Next, we assume that $P(k)$ is true for some positive integer k, i.e.

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2} \; .$$

and we want to use this to prove $P(k+1)$, i.e.

$$1 + 2 + 3 + \cdots + (k+1) = \frac{(k+1)(k+2)}{2} \; .$$

Taking the LHS and using $P(k)$,

$$\begin{aligned} 1 + 2 + 3 + \cdots + (k+1) &= (1 + 2 + 3 + \cdots + k) + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

and thus $P(k+1)$ is true. This completes the proof.

Problem 2. Find a formula for the sum of the first n odd numbers.

Solution. Note that this time we are not told the formula that we have to prove; we have to find it ourselves! Let's try some small numbers and see if a pattern emerges:

$$1 = 1; \quad 1 + 3 = 4; \quad 1 + 3 + 5 = 9;$$
$$1 + 3 + 5 + 7 = 16; \quad 1 + 3 + 5 + 7 + 9 = 25;$$

We conjecture (guess) that the sum of the first n odd numbers is equal to n^2. Now let's prove this proposition using the principle of induction; call it $P(n)$.
Our statement $P(n)$ is that

$$1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2 \ .$$

First we prove the base case $P(1)$, i.e.

$$1 = 1^2$$

This is certainly true. Now we assume that $P(k)$ is true, i.e.

$$1 + 3 + 5 + 7 + \cdots + (2k - 1) = k^2 \ .$$

and consider $P(k + 1)$:

$$1 + 3 + 5 + 7 + \cdots + (2k + 1) = (k + 1)^2 \ .$$

Taking the LHS and using $P(k)$,

$$\begin{aligned} 1+3+5+\cdots+(2k+1) &= (1+3+5+\cdots+(2k-1))+(2k+1) \\ &= k^2+(2k+1) \\ &= (k+1)^2 . \end{aligned}$$

and thus $P(k+1)$ is true. This completes the proof.

Note. Find a US flag and see if you can use it to prove this result another way which does not require induction. [**Hint:** Look at the stars!]

Exercise 1. Show that for all $n \geq 1$,

$$1^2+3^2+5^2+\cdots+(2n-1)^2=\frac{n(4n^2-1)}{3} .$$

Exercise 2. Show that for all $n \geq 1$, we have $f(n)=g(n)$, where

$$f(n)=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2n-1}-\frac{1}{2n}$$

and

$$g(n)=\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{2n} .$$

Problem 3. Show that 6 divides $8^n - 2^n$ for every positive integer n.

Solution. We will use induction. First we prove the base case $n = 1$, i.e. that 6 divides $8^1 - 2^1 = 6$; this is certainly true.
Next assume that proposition holds for some positive integer k, i.e. 6 divides $8^k - 2^k$. Let's examine $8^{k+1} - 2^{k+1}$:

$$\begin{aligned} 8^{k+1} - 2^{k+1} &= 8 \cdot 8^k - 2 \cdot 2^k \\ &= 6 \cdot 8^k + 2 \cdot 8^k - 2 \cdot 2^k \\ &= 6 \cdot 8^k + 2 \cdot \left(8^k - 2^k\right) \ . \end{aligned}$$

Now since 6 divides $8^k - 2^k$ (by assumption), and 6 certainly divides $6 \cdot 8^k$, it follows that 6 divides $8^{k+1} - 2^{k+1}$. Therefore by the principle of induction, 6 divides $8^n - 2^n$ for every positive integer n.

Exercise 3. For every $n \geq 1$, define

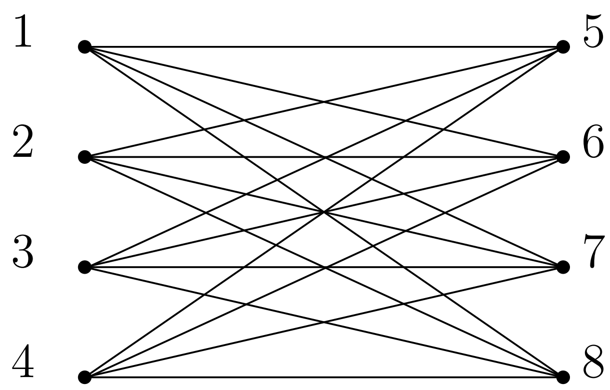
$$S(n) = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30} \ .$$

Show that $S(n)$ is an integer for every $n \geq 1$.

Problem 4. $2n$ points are given in space, where $n \geq 2$. Altogether n^2+1 line segments ('edges') are drawn between these points. Show that there is at least one set of three points which are joined pairwise by line segments (i.e. show that there exists a *triangle*).

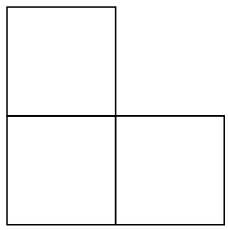
Solution. The proposition (let's call it $P(n)$) holds for $n = 2$ (why?). Assume that the proposition $P(n)$ is true for $n = k$, i.e. that if $2k$ points are joined together by k^2+1 edges, there must exist a triangle. Now consider $P(k+1)$: here we have $2(k+1) = 2k+2$ points, which are connected by $(k+1)^2+1 = k^2+2k+2$ edges. Take a pair of points A, B which are joined by an edge (there must be such a pair, otherwise there are no edges connecting any of the points!). The remaining $2k$ points form a set which we will call $\mathcal{S}$. Let's focus on the set $\mathcal{S}$ for the moment. If there were at least k^2+1 edges in $\mathcal{S}$, then there would have to be a triangle in here (using the $P(k)$ assumption). Of course there could be $\leq k^2$ edges in $\mathcal{S}$; let's suppose this is the case. But if this were true, it would mean that there are at least $2k+2$ edges in the other part of the graph, i.e. connecting A and B to each other and to the points in $\mathcal{S}$. Discounting the edge AB gives at least $2k+1$ edges which connect from A or B into $\mathcal{S}$. But we notice that if P is a point in $\mathcal{S}$, then P can be connected either to A or B, but not both (or a triangle PAB would be formed!). Therefore the maximum number of edges connecting from A or B into $\mathcal{S}$ (without forming a triangle) is $2k$. This contradiction proves that $P(k+1)$ must be true.

Note. If we have $2n$ points and *exactly* n^2 edges, it is possible to *avoid* making a triangle. This is done by breaking the set of points into two subsets $\mathcal{X}$ and $\mathcal{Y}$ which contain n points each, then connecting every point in $\mathcal{X}$ to every point in $\mathcal{Y}$. This is illustrated below for the case $n = 4$.



Exercise 4.

Let n be a positive integer. Prove that if one square of a $2^n \times 2^n$ chessboard is removed, the remaining board can be tiled with 3-square tiles of the following shape:



Exercise 5.

Let $f(n)$ be the number of regions which are formed by n lines in the plane, where no two lines are parallel and no three meet at a single point (e.g. $f(1) = 2$; $f(2) = 4$; etc.). Find a formula for $f(n)$.

Exercise 6. Every road in Uniland is one-way. Every pair of cities is connected by exactly one direct road. Show that there exists a city which can be reached from every other city either directly of via at most one other city.

Pólya's Paradox:

A common way (in 1950, at least!) of expressing that something is out of the ordinary is "*That's* a horse of a different color!" The famous mathematician George Pólya gave the following proof that "all horses are the same color", which works by the principle of induction:

Proposition $P(n)$: Suppose we have n horses. Then all n horses are the same colour.

Base case: $n = 1$; if there is only one horse, there is only one colour.

Inductive step: Assume that $P(k)$ is true, i.e. that for any set of k horses, there is only one color. Now look at any set of $k + 1$ horses; call this $\{H_1, H_2, H_3, \cdots, H_k, H_{k+1}\}$. Consider the sets $\{H_1, H_2, H_3, \cdots, H_k\}$ and $\{H_2, H_3, H_4, ..., H_{k+1}\}$. Each is a set of only k horses, therefore within each there is only one colour. But the two sets overlap, so there must be only one colour among all $k + 1$ horses.

The flaw is that when $k = 2$ the inductive step doesn't work, because the statement that "the two sets overlap" is false.