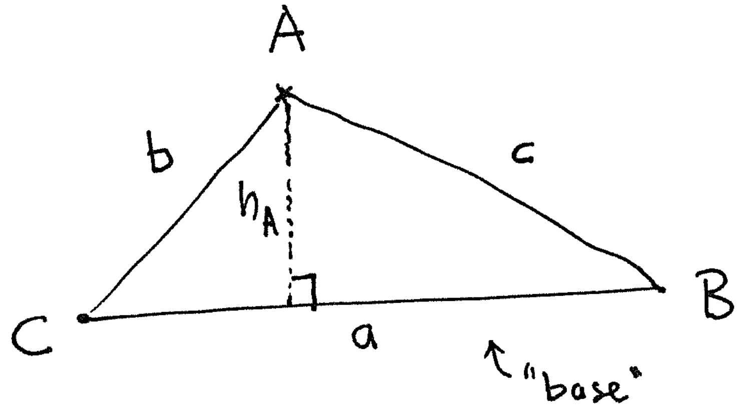
MATHEMATICAL ENRICHMENT

Jan 19^{th} 2019 (11:45 – 1pm)

KEVIN HUTCHINSON : Geometry



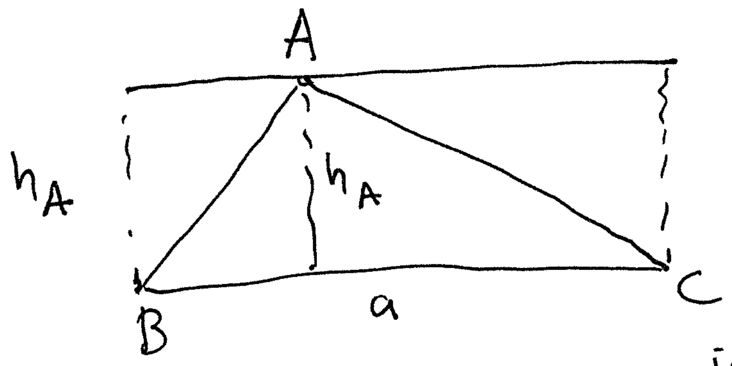
Triangles

$AB = c$. $BC = a$

h_A = "altitude at A"

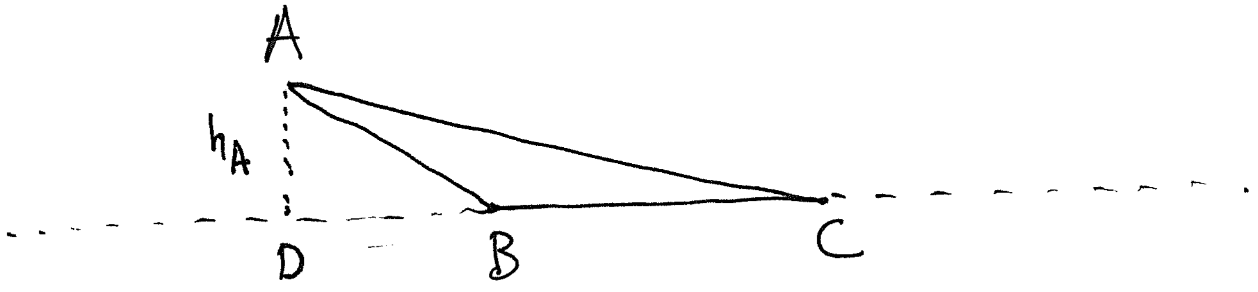
Area of ΔABC is $[ABC] = \frac{1}{2} \times$ base $\times$ perp. height.

So $\boxed{[ABC] = \frac{1}{2} a \cdot h_A}$



area of rectangle is $a \cdot h_A$.
area of triangle is ½ of this.

<u>Caution:</u> What if the altitude is external?



$$[ABC] = \underset{\frac{1}{2} h_A \cdot DC}{\underset{\shortparallel}{[ADC]}} - \underset{\frac{1}{2} h_A \cdot DB.}{\underset{\shortparallel}{[ADB]}}$$

$$= \frac{1}{2} h_A \cdot (DC - DB)$$

$$= \frac{1}{2} h_A \cdot BC = \frac{1}{2} a h_A$$ <u>Still valid!</u>

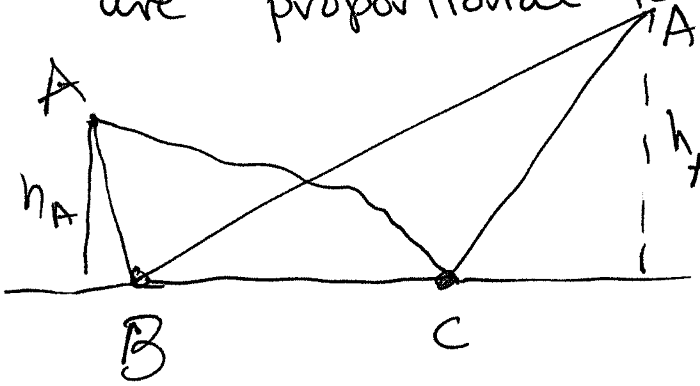
<u>"Corollary"</u> ABC any triangle.

$$[ABC] = \frac{1}{2} a \cdot h_A = \frac{1}{2} b \cdot h_B = \frac{1}{2} c \cdot h_C$$

i.e. direct logical consequence

<u>Some more</u> corollaries.

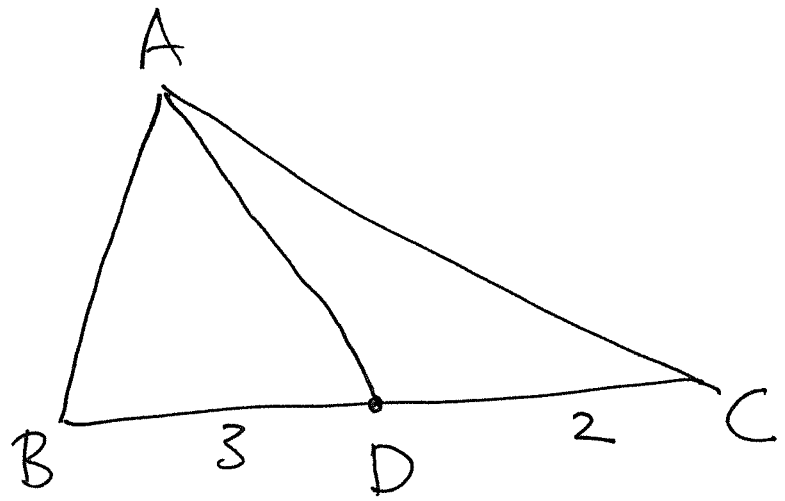
1. Areas of triangles with equal bases are proportional to their altitudes:



$$\frac{[A'BC]}{[ABC]} = \frac{h_{A'}}{h_A}$$

2. Areas of Δs with equal altitude are proportional to their bases.

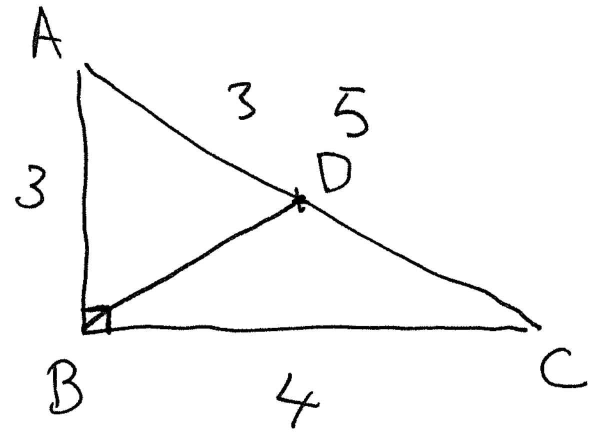
$$\frac{[ABC]}{[A'B'C']} = \frac{BC}{B'C'} \quad \text{if } h_A = h_{A'}$$



Eg $\frac{[ABD]}{[ADC]} = \frac{3}{2}$: ie. $[ABD] = \frac{3}{2}[ADC]$

Sim. $[ABD] = \frac{3}{5}[ABC]$

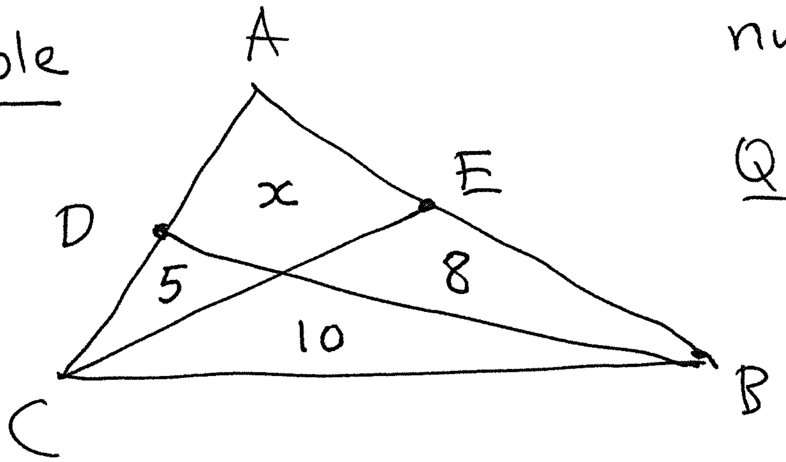
<u>Example</u>



What's $[ABD]$? $[ABD] = \frac{3}{5}[ABC]$

$= \frac{3}{5} \cdot \left(\frac{1}{2} 3 \cdot 4\right) = \frac{18}{5}$

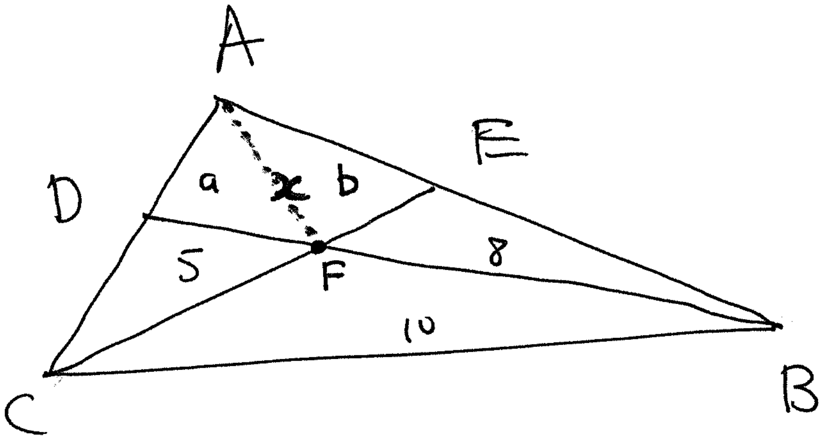
Example



numbers are areas.

Q: $x = ?$

Solution



$x = a + b.$

$$\frac{[ABD]}{[DBC]} = \frac{AD}{DC} = \frac{[AFD]}{[DFC]}$$

$$\boxed{\frac{8+a+b}{15} = \frac{a}{5}} \quad (1)$$

Similarly

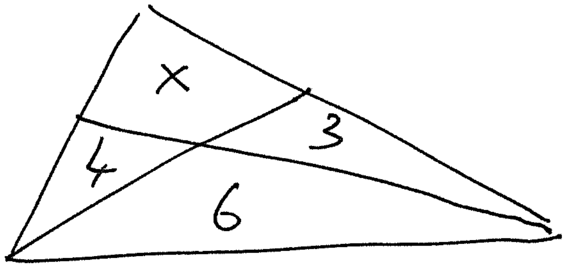
$$\frac{[ACE]}{[ECB]} = \frac{AE}{EB} = \frac{[AFE]}{[EFB]}$$

$$\boxed{\frac{5+a+b}{18} = \frac{b}{8}} \quad (2)$$

Solve to find
$a = 10$, $b = 12$

So $\boxed{x = 22}$

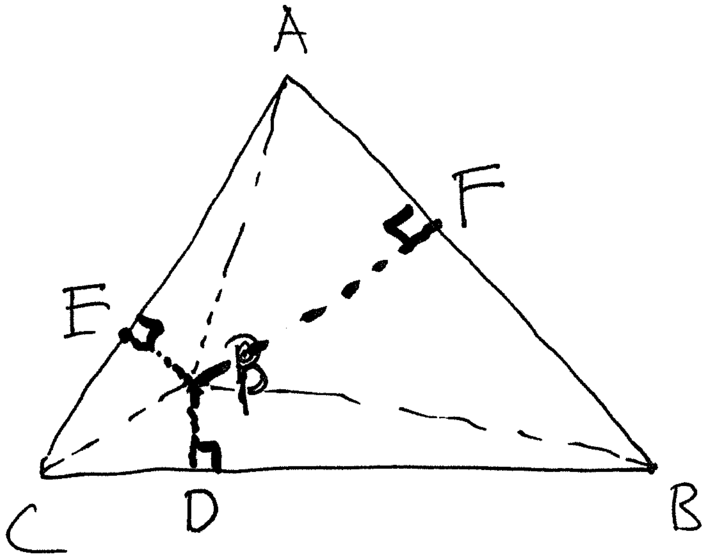
Exercise



$x = ?$

Viviani's Theorem

Let P be any point inside the equilateral triangle ABC



$(AB = BC = CA)$

Let PE, PF, PD be the perpendiculars from P to the sides.

Then $PE + PF + PD = h$, the altitude of ABC

Solution

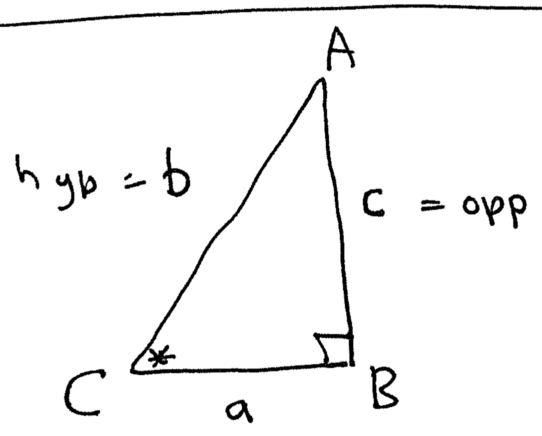
$$[ABC] = [APC] + [APB] + [BPC]$$

$$\frac{1}{2} h_A \cdot BC = \frac{1}{2} \cdot PE \cdot AC + \frac{1}{2} PF \cdot AB + \frac{1}{2} PD \cdot BC$$

$$= \frac{1}{2}(PE + PF + PD) \cdot BC$$

$$\Rightarrow \quad h_A = PE + PF + PD.$$

Recall

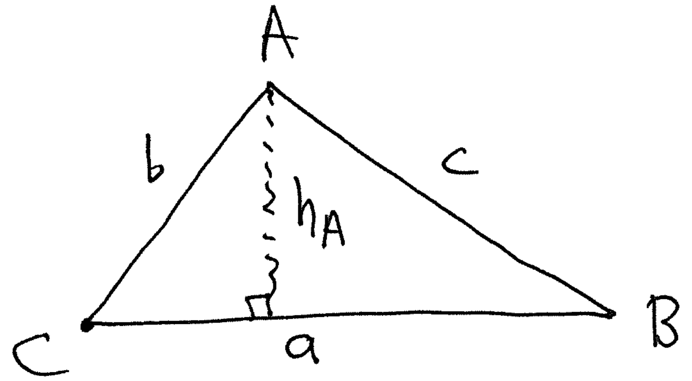


$$\sin C = \frac{opp}{hyp}$$

Theorem $[ABC] = \frac{1}{2} a b \sin C$

for any triangle ABC.

Proof:



$$[ABC] = \frac{1}{2} a \cdot h_A$$

$$\sin C = \frac{h_A}{b} \Rightarrow \boxed{h_A = b \sin C}$$

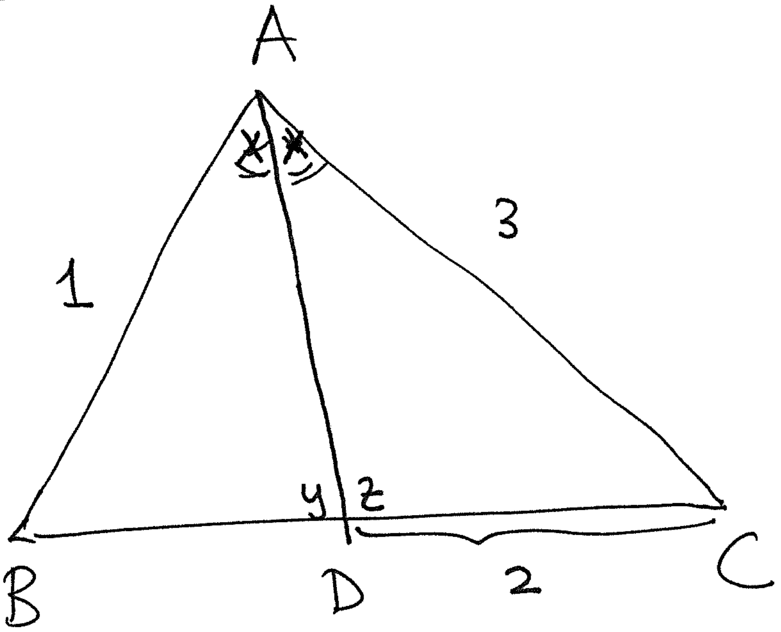
$$= \frac{1}{2} a \cdot b \cdot \sin C$$

Note $\quad h_A = b\sin C = c\sin B$

$$\Rightarrow \quad \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\boxed{\text{Sine Rule} \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{for any triangle } ABC}$$

Example



Q: What is BC?

Sine Rule:

$$\frac{3}{\sin z} = \frac{2}{\sin x} \quad \text{and} \quad \frac{BD}{\sin x} = \frac{1}{\sin y}$$

$$y + z = 180° \Rightarrow \sin y = \sin z$$

$$\frac{2}{3} = \frac{\sin x}{\sin z} = \frac{BD}{1} \quad \Rightarrow \quad \boxed{BD = \frac{2}{3}}$$

Question 1

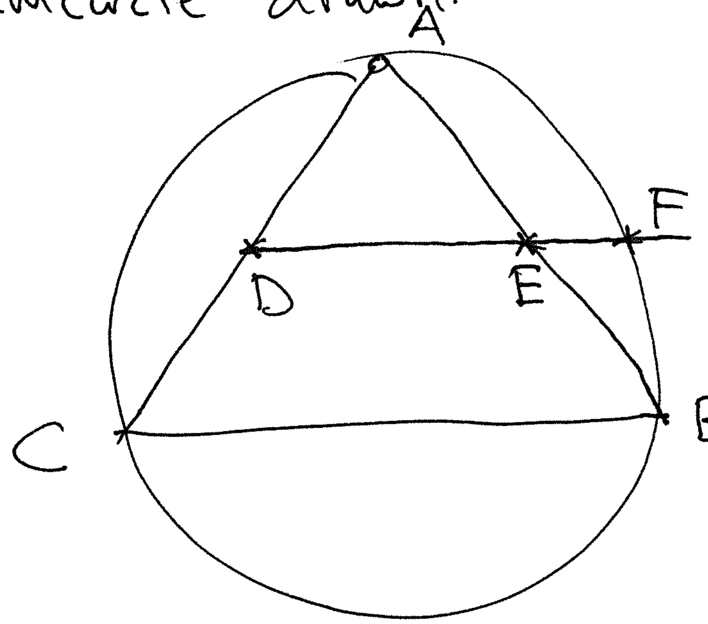
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} =$$

Some important geometric invariant of the triangle.

What? — Find out.

Question 2

ABC equilateral triangle with circumcircle drawn.



D, E are midpoint of AC, AB. DE (extended) meets circle at F.

Show that

$$\frac{DF}{DE} = \frac{DE}{EF}$$

(ie. "golden section")
