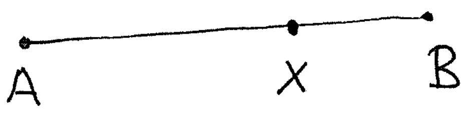
Mathematical Enrichment   March 2nd, 2019

Geometry   Kevin Hutchinson

From last time ...

"Golden section" or "ratio" or "mean"

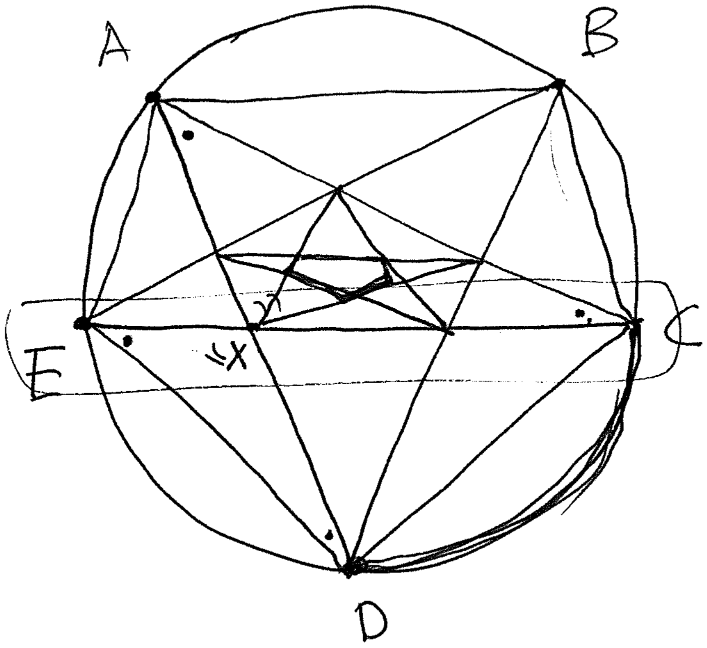


$$\frac{AX}{AB} = \frac{XB}{AX} = \frac{AB - AX}{AX} = \frac{AB}{AX} - 1$$

$\parallel$ r satisfies $\quad r = \frac{1}{r} - 1$

So $\quad r^2 + r - 1 = 0$

Solve $\quad \boxed{r = \frac{-1+\sqrt{5}}{2}}$



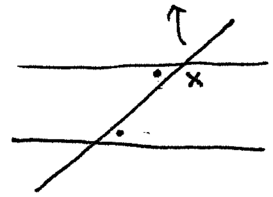
$EC \parallel AB$

$AD \parallel BC$

$\therefore$ $ABCX$ is a $\parallel$ogram with 4 equal sides.

$\angle DEC = \angle DAC$ (same arc)

$\angle DEC = \angle ECA = \angle EDA$



alternating angles

So $AXC \sim EXD$

"similar"

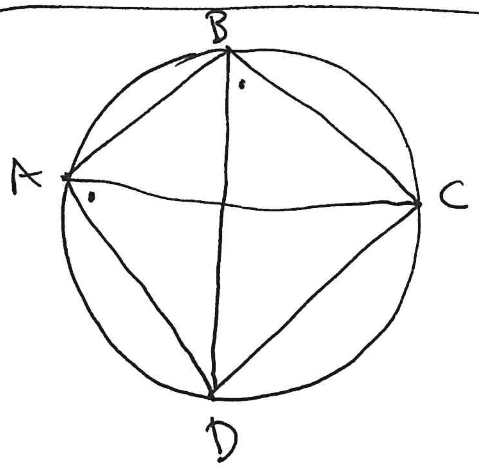
$$\Rightarrow \quad \frac{AC}{ED} = \frac{AX}{EX}.$$

But $AC = EC$

$ED = AB = CX$

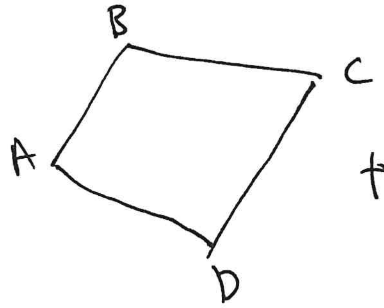
$AX = CX$.

$$\therefore \quad \frac{EC}{CX} = \frac{CX}{EX} \quad \& \quad CX = AB$$



$\angle B + \angle D = 180^\circ$

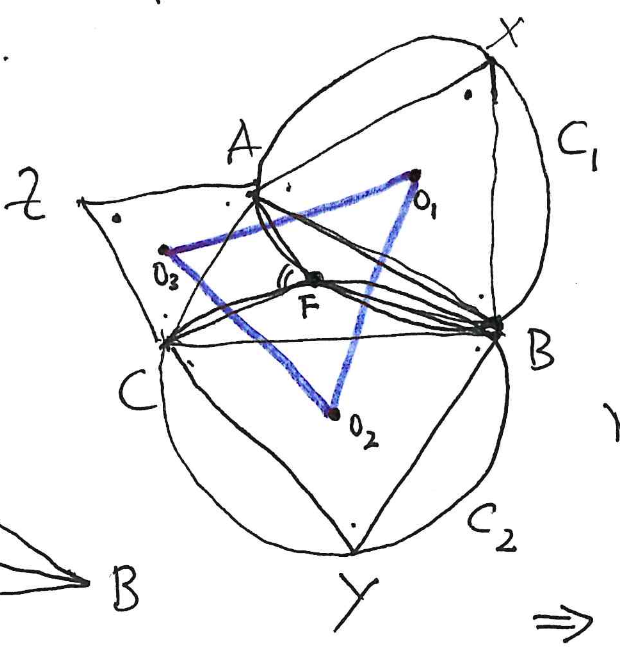
Conversely, ABCD convex quadrilateral



If $\angle B + \angle D = 180^\circ$ then it is cyclic.

ABC a Δ, all angles $< 120^\circ$

Construct an equilateral triangle externally on each side.



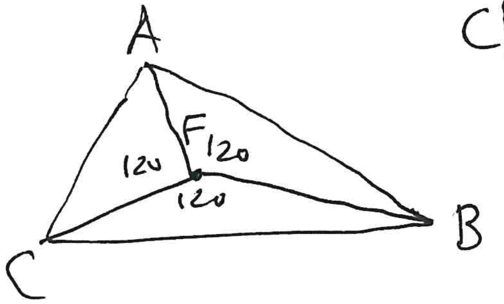
Circumcircles C_1 and C_2 of AXB and CYB meet at B and F.

F is also on C_3:

Note that

$\angle AFB = 120^\circ$

$(\angle AFB + \angle X = 180^\circ)$ where $\angle X = 60$



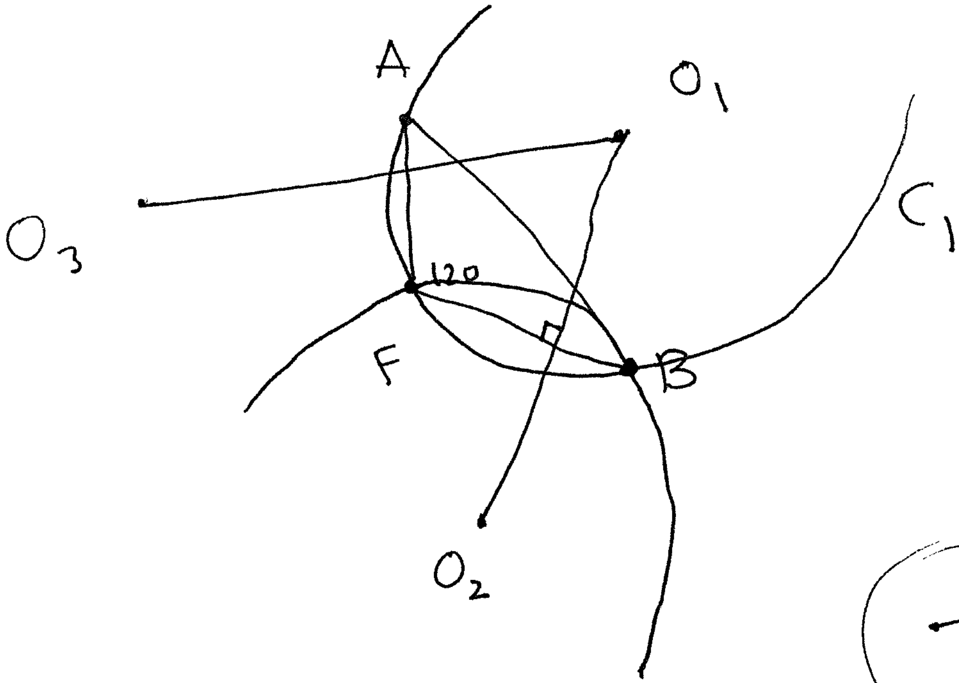
$\Rightarrow \quad \angle CFB = 120^\circ$

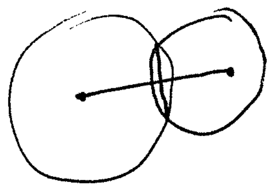
So $\angle CFA = 120^\circ$ also.

$\therefore \quad \angle CFA + \angle Z = 120^\circ + 60^\circ = 180^\circ$

$\Rightarrow \quad CFAZ$ is cyclic quad

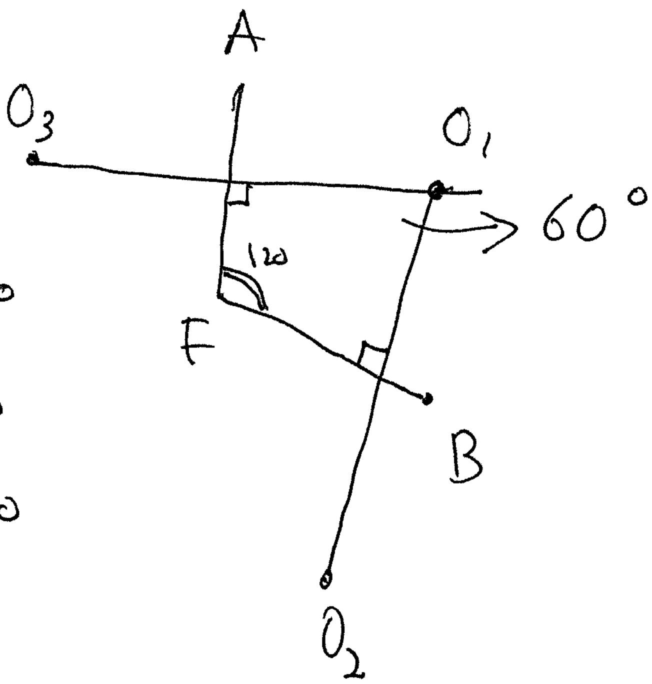
$\Rightarrow \quad F$ lies on circumcircle C_3 of CZA





$O_1O_2 \perp FB$

$O_1O_3 \perp FA$



$\Rightarrow \quad \angle O_2O_1O_3 = 60^\circ$

Likewise $\angle O_1O_2O_3 = 60^\circ$

$\angle O_1O_3O_2 = 60^\circ$

$\Rightarrow \quad O_1O_2O_3$ is an <u>equilateral triangle</u>.

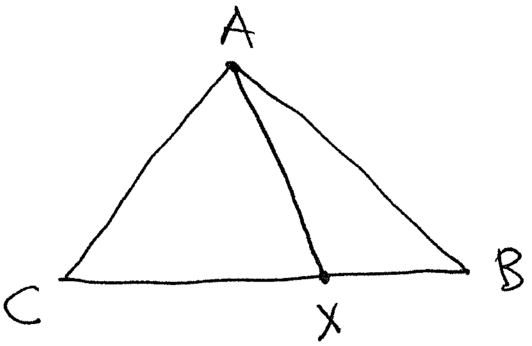
"<u>Napoleon's Theorem</u>"

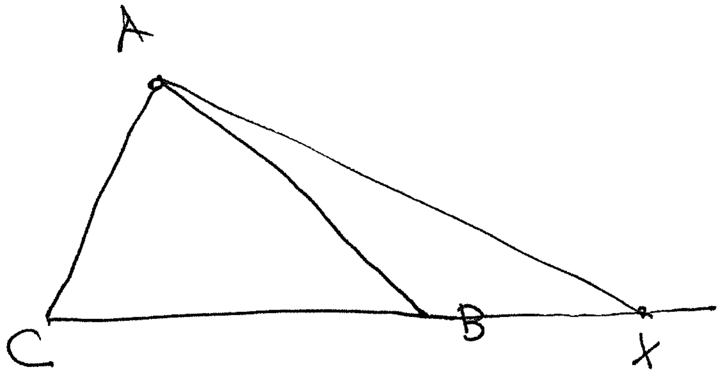
Ceva's Theorem

Giovanni Ceva 1678

Let ABC be any triangle.

A "cevian" is a line joining a vertex to the opposite side



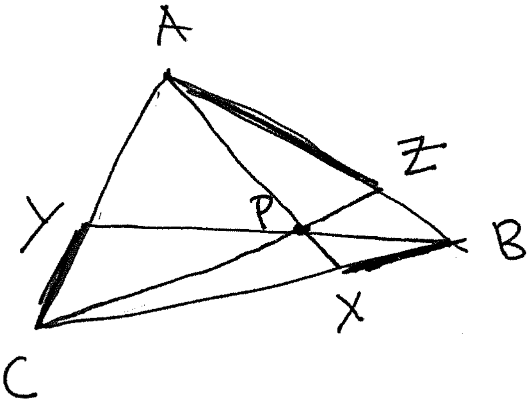


AX is a cevian (internal)

AX is an external cevian.

Theorem If 3 cevians AX, BY, CZ are concurrent, then

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1$$



Proof:

$$\frac{BX}{XC} = \frac{[XAB]}{[CAX]} = \frac{[XPB]}{[CPX]} \overset{(*)}{=} \frac{[XAB]-[XPB]}{[CAX]-[CPX]}$$

$$= \frac{[BPA]}{[CPA]}$$

Check (*)

If $\frac{x}{y} = \frac{z}{w}$ then

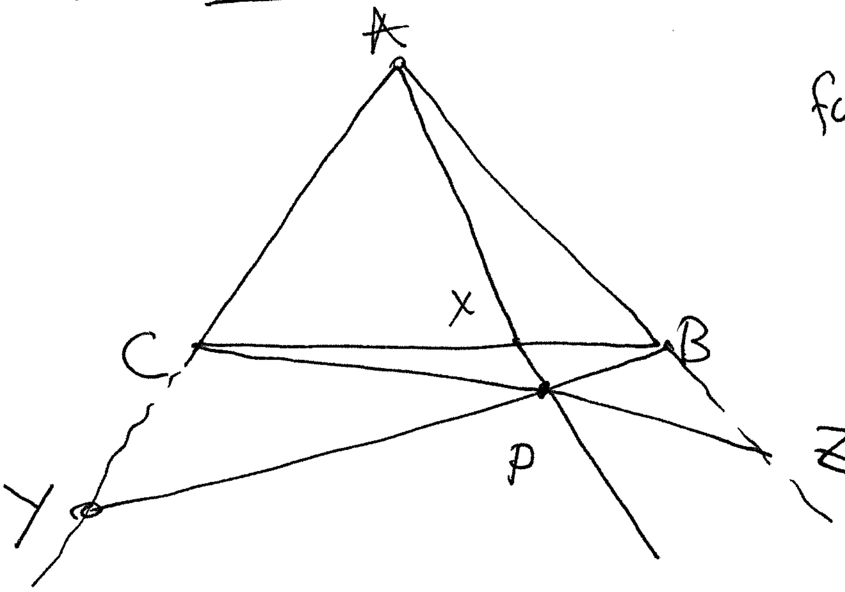
$$\frac{x}{y} = \frac{x-z}{y-w}$$

By the same reasoning

$$\frac{CY}{YA} = \frac{[CPB]}{[APB]}, \quad \frac{AZ}{ZB} = \frac{[CPA]}{[BPC]}$$

So $$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = \cancel{\frac{[BPA]}{[CPA]}} \cdot \cancel{\frac{[BPC]}{[APB]}} \cdot \cancel{\frac{[CPA]}{[BPC]}} = 1$$

True for external cevians



Same idea works for the proof: Exercise

Converse to Ceva's Theorem

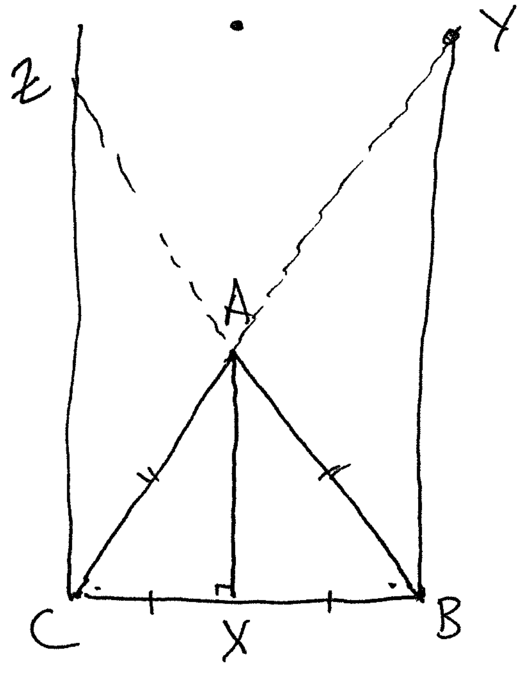
If AX, BY, CZ are cevians and if

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1$$ then the

three lines are concurrent (or all three are parallel to each other!)

(**).

(**)



$$\frac{BX}{XC}\,\frac{CY}{AY}\cdot\frac{AZ}{ZB} = 1 \quad ✓$$

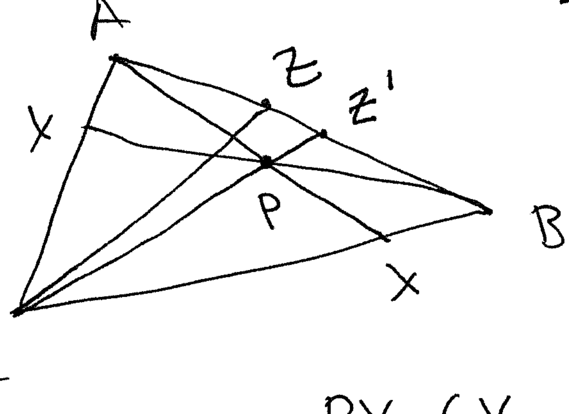
$$\frac{BX}{XC} = 1$$

(* In projective geometry the 3 lines are concurrent at a "point at infinity").

Proof of (Converse to) Ceva's Theorem

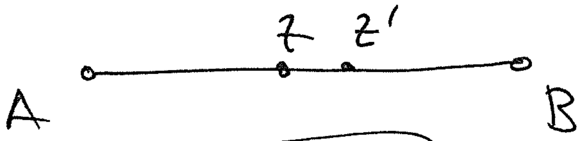
Suppose, without loss, that AX and BY have a point P in common.

Let CZ' be the cevian through P



By hypothesis $\quad \frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB} = 1$

By Ceva's Thm $\quad \frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ'}{Z'B} = 1$



(one case: Both Z, Z' internal)

(external cevians: same reasoning)

$$\boxed{\frac{AZ}{ZB} = \frac{AZ'}{Z'B}}$$

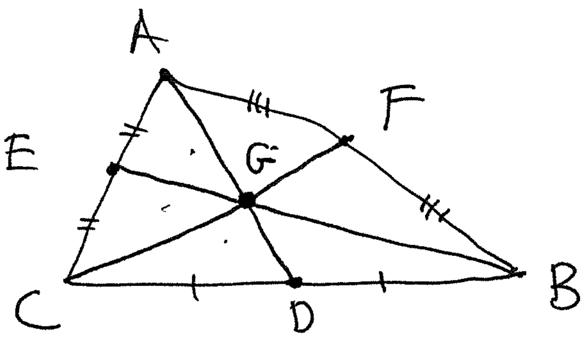
If $AZ < AZ'$ then $ZB > Z'B$

$$\Rightarrow \quad \frac{AZ}{ZB} < \frac{AZ'}{Z'B} \quad \rightarrow\leftarrow$$

Similarly if $AZ > AZ'$ then $\dfrac{AZ}{ZB} > \dfrac{AZ'}{Z'B}$ $\rightarrow\leftarrow$

So $AZ = AZ'$ and $Z = Z'$

Examples (1) The medians (lines joining vertex to middle of opposite side) are concurrent.



G = centroid or centre of gravity.

Exercise Show all 6 smaller Δs have equal area.

Exercise F = "Fermat point" of ΔABC.

Let P be any other point inside ABC.

Show that $PA + PB + PC > FA + FB + FC$.

Exercise Let AX, BY, CZ be internal cevians which are concurrent.

Prove that YZ is parallel to BC if and only if X is the midpoint of BC

Exercise Show that the lines joining the vertices of a Δ to the points of tangency of the incircle with the opposite side are concurrent.