MATHEMATICAL ENRICHMENT / OLYMPIAD TRAINING U.C.D.

5 FEBRUARY 2022 PROFESSOR GARY MCGUIRE

INEQUALITIES 1

BASIC INEQUALITY : $x^2 \geq 0$ and equality holds if and only if $x = 0$

Example Show $x^2+y^2+z^2 \geq xy + yz + zx$

Solution:

$$(x-y)^2 \geq 0 \Rightarrow x^2 - 2xy + y^2 \geq 0$$
$$(y-z)^2 \geq 0 \Rightarrow y^2 - 2yz + z^2 \geq 0$$
$$(z-x)^2 \geq 0 \Rightarrow z^2 - 2zx + x^2 \geq 0$$

Add: $2x^2 + 2y^2 + 2z^2 - 2xy - 2yz - 2zx \geq 0$

Rearrange and divide by 2.

Note Equality holds if and only if $x = y = z$

Example Show $\dfrac{a^2}{a+b} \geq \dfrac{3a-b}{4}$ where $a>0, b>0$

Solution: $(a-b)^2 \geq 0$

$$\Rightarrow a^2 - 2ab + b^2 \geq 0$$
$$a^2 \geq 2ab - b^2$$

add $3a^2$ $\quad 4a^2 \geq 3a^2 + 2ab - b^2 = (3a-b)(a+b)$

Divide by $a+b$ $\quad \dfrac{4a^2}{a+b} \geq 3a-b$

Then divide by 4.

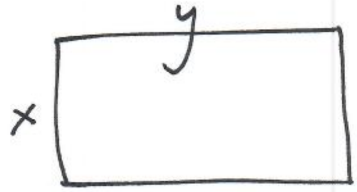
Equality holds if and only if $a=b$

Example Show $\dfrac{a^2}{a+b} + \dfrac{b^2}{b+c} + \dfrac{c^2}{c+a} \geq \dfrac{a+b+c}{2}$ where $a,b,c>0$

Solution: By previous example

L.H.S. is $\geq \dfrac{3a-b}{4} + \dfrac{3b-c}{4} + \dfrac{3c-a}{4}$

$$= \frac{3a-b+3b-c+3c-a}{4}$$
$$= \frac{2a+2b+2c}{4} = \frac{a+b+c}{2}$$



Suppose $x+y$ is fixed. $\begin{pmatrix} x>0 \\ y>0 \end{pmatrix}$

The area is largest when $x=y$.

<u>proof</u>: use $(x-y)^2 \geq 0$ and equality holds iff $x=y$

$$x^2 - 2xy + y^2 \geq 0$$

Add $4xy$: $\quad x^2 + 2xy + y^2 \geq 4xy \quad$ ''

$$(x+y)^2 \geq 4xy \quad ''$$

$$\underbrace{\left(\frac{x+y}{2}\right)^2}_{\text{fixed.}} \geq xy \quad ''$$

So xy is a maximum when $x=y$.

The ARITHMETIC MEAN of x, y is $\frac{x+y}{2}$

Then GEOMETRIC MEAN of x,y is $\sqrt{xy}$

The AM–GM inequality is AM $\geq$ GM

$$\boxed{\frac{x+y}{2} \geq \sqrt{xy}}$$

$\begin{pmatrix} x>0 \\ y>0 \end{pmatrix}$ equality holds iff $x=y$

<u>Example</u> If $x>0$ show $x + \frac{1}{x} \geq 2$. (and equality holds iff $x=1$)

<u>Solution</u>: use AM–GM inequality with $y = \frac{1}{x}$

$$\frac{x + \frac{1}{x}}{2} \geq \sqrt{x \cdot \frac{1}{x}} = 1$$

Multiply by 2 $\quad x + \frac{1}{x} \geq 2$. $\quad$ Eq. holds iff $x = \frac{1}{x}$, $x^2 = 1$, $x = 1$

<u>Note</u> Same solution shows

$$\frac{a}{b} + \frac{b}{a} \geq 2$$

<u>Example</u> For $a, b, c > 0$ show

$$(a+b)(b+c)(a+c) \geq 8abc.$$

Solution: By the AM-GM inequality

$$\frac{a+b}{2} \geq \sqrt{ab}, \quad \frac{b+c}{2} \geq \sqrt{bc}, \quad \frac{a+c}{2} \geq \sqrt{ac}$$

Multiply these

$$\left(\frac{a+b}{2}\right)\left(\frac{b+c}{2}\right)\left(\frac{a+c}{2}\right) \geq \sqrt{(ab)(bc)(ac)} = \sqrt{a^2b^2c^2} = abc$$

Multiply by 8.

The AM-GM inequality for n numbers is

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

equality holds iff $a_1 = a_2 = \cdots = a_n$.

<u>Example</u> show $\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \geq 3$ where $x > 0$, $y > 0$, $z > 0$

<u>Solution</u> use AM-GM inequality with $n = 3$

$$\frac{\frac{x}{y} + \frac{y}{z} + \frac{z}{x}}{3} \geq \sqrt[3]{\left(\frac{x}{y}\right)\left(\frac{y}{z}\right)\left(\frac{z}{x}\right)} = 1.$$

Multibly by 3.

<u>Example</u> Find the minimum of $\frac{6x}{y} + \frac{12y}{z} + \frac{3z}{x}$

Apply AM-GM

$$\frac{6x}{y} + \frac{12y}{z} + \frac{3z}{x} \geq 3\sqrt[3]{\left(\frac{6x}{y}\right)\left(\frac{12y}{z}\right)\left(\frac{3z}{x}\right)}$$

$$= 3 \cdot \sqrt[3]{216}$$

$$= (3)\cdot(6) = 18.$$

<u>Example</u> Show $x^6 + y^6 + 4 \geq 6xy$.

Apply AM–GM inequality with $x^6, y^6, 1, 1, 1, 1$

$$\Rightarrow \quad \frac{x^6 + y^6 + 1 + 1 + 1 + 1}{6} \geq \sqrt[6]{(x^6)(y^6)(1)(1)(1)(1)}$$

$$= xy.$$

<u>Example</u> Minimize $x^2 + \frac{a}{x}$ where a is fixed.

<u>Solution</u> Apply AM–GM inequality with $n = 3$

to $x^2 + \frac{1}{2} \cdot \frac{a}{x} + \frac{1}{2} \cdot \frac{a}{x}$.

Get $x^2 + \frac{1}{2}\frac{a}{x} + \frac{a}{2x} \geq 3\sqrt[3]{(x^2)\left(\frac{a}{2x}\right)\left(\frac{a}{2x}\right)}$

$$= 3\sqrt[3]{a^2/4}$$

This is the minimum. $\quad = 3\left(\frac{a}{2}\right)^{2/3}$