Maths Enrichment
Functional Equations

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1 Introduction

1.1 What is a Functional Equation?

1.2 IMO 2019, Bath. Question 1

Let $\mathbb{Z}$ be the set of integers. Determine all functions $f : \mathbb{Z} \mapsto \mathbb{Z}$ such that, for all integers a and b,

$$f(2a) + 2f(b) = f(f(a+b)) \tag{1}$$

2 Cauchy's Functional Equation

$$f(a+b) = f(a) + f(b) \tag{2}$$

2.1 Cauchy's Equation over Non-negative Integers

Put $a = 0$ in equation 2

$$f(b) = f(0) + f(b)$$

Therefore $f(0) = 0$.

Suppose that for some positive integer a the following inductive hypothesis holds:

$$f(a) = af(1) \tag{3}$$

Then, putting $b = 1$ in equation 2

$$\begin{aligned} f(a+1) &= f(a) + f(1) \\ &= af(1) + f(1) \\ &= (a+1)f(1) \end{aligned}$$

So the claimed equation 3 also holds for $a + 1$. As the relation is trivially true when $a = 0$, then it holds for all $a \in \mathbb{Z}^+$.

We need finally to check that this indeed satisfies the original functional equation. This plainly works as for arbitrary non-negative integers a, b and $f(1)$ we have:

$$f(a+b) = (a+b)f(1) = af(1) + bf(1) = f(a) + f(b)$$

2.2 Cauchy's Equation over Integers

For $0 \leq a \in \mathbb{Z}$ we already have the solution in equation 3.

For $a \geq 0$, set $b = -a$ in equation 2, to give:

$$0 = f(a - a) = f(a) + f(-a)$$

Hence equation 3 holds for all $a \in \mathbb{Z}$.

2.3 Cauchy's Equation over Rationals

Suppose $\mathbb{Q} \ni a = \frac{p}{q}$ with p and q integers and $q > 0$.

Then by a similar inductive process to our proof of equation 3, we have:

$$f(p) = f\left(q\frac{p}{q}\right) = qf\left(\frac{p}{q}\right)$$

As $p \in \mathbb{Z}$ we also have $f(p) = pf(1)$ and so:

$$f\left(\frac{p}{q}\right) = \frac{p}{q}f(1)$$

We have then shown that equation 3 holds for all $a \in \mathbb{Q}$.

2.4 Cauchy's Equation over Reals

Let us now consider solutions to equation 2 for $a, b \in \mathbb{R}$. The reals include the rational numbers $\mathbb{Q}$ but also numbers such as $\sqrt{2}$ and π that cannot be expressed as fractions.

Any function of the form $f(x) = cx$ for $c \in \mathbb{R}$ satisfies equation 2. But are these the only solutions? To put this differently, we know that any solution much have (for some $c \in \mathbb{R}$)

$$f(x) = cx; x \in \mathbb{Q}$$

But can we be sure the same multiple c works for $x \notin \mathbb{Q}$?

To take a specific example, let us try $x = \sqrt{2}$. We take the opportunity to practice some binomial expansions. We note first the trivial result that for integers $n \geq 0$:

$$(1 - \sqrt{2})^n = \frac{1}{2}\left[(1 + \sqrt{2})^n + (1 - \sqrt{2})^n\right] - \frac{1}{2}\left[(1 + \sqrt{2})^n - (1 - \sqrt{2})^n\right]$$

Looking at the binomial expansions of the first square brackets, all the odd powers of $\sqrt{2}$ cancel out, so the square brackets in total are an even integer. In the same way, expanding binomial terms in the second square bracket, all the even powers of $\sqrt{2}$ cancel out, so the expression is twice an integer multiplied by $\sqrt{2}$.

Thus, we can write the power on the left as an integer minus an integer multiple of $\sqrt{2}$:

$$(1 - \sqrt{2})^n = \frac{1}{2}\left[(1 + \sqrt{2})^n + (1 - \sqrt{2})^n\right] - \frac{1}{2\sqrt{2}}\left[(1 + \sqrt{2})^n - (1 - \sqrt{2})^n\right]\sqrt{2}$$

Applying the function f to each side and using equation 2 repeatedly, we have:

$$\begin{aligned} f\left[(1-\sqrt{2})^n\right] &= \frac{1}{2}\left[(1+\sqrt{2})^n + (1-\sqrt{2})^n\right] f(1) \\ &- \frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^n - (1-\sqrt{2})^n\right] f(\sqrt{2}) \\ &= (1-\sqrt{2})^n \left[\frac{f(1)}{2} + \frac{f(\sqrt{2})}{2\sqrt{2}}\right] \\ &+ (\sqrt{2}-1)^{-n} \left[\frac{f(1)}{2} - \frac{f(\sqrt{2})}{2\sqrt{2}}\right] \end{aligned}$$

In the last line we have used the difference of two squares:

$$(\sqrt{2}+1)(\sqrt{2}-1) = \sqrt{2}^2 - 1^2 = 1$$

Combining this with the rational case, we have two distinct behaviours:

$$f(a) = \begin{cases} af(1) & a \in \mathbb{Q} \\ a\left[\frac{f(1)}{2} + \frac{f(\sqrt{2})}{2\sqrt{2}}\right] + \frac{1}{|a|}\left[\frac{f(1)}{2} - \frac{f(\sqrt{2})}{2\sqrt{2}}\right] & a = (1-\sqrt{2})^n \end{cases}$$

Of course, not all real a fall into one of these categories, but just looking at these points, we have a function that looks like a combination of straight lines and hyperbolas, unless $f(\sqrt{2}) = \sqrt{2}f(1)$ in which case the reciprocal terms vanish.

This tells us that *if* there is a solution with $f(\sqrt{2}) \neq \sqrt{2}f(1)$ then that function is badly behaved; certainly not continuous or even bounded at zero, nor indeed anywhere else. It turns out (you won't be able to prove this without a lot more advanced maths) that these solutions do exist, and are called Hamel functions.

If we want the solution to equation 2 to be continuous over the reals, then the Hamel functions are excluded and the only solutions are linear.

3 A 2020 Problem

3.1 Problem Statement

Find all functions $f : \mathbb{Z} \mapsto \mathbb{Z}$ such that, for all $n \in \mathbb{Z}$:

$$f(f(n)) = n + 2020 \tag{4}$$

3.2 A Trivial Solution

$$f(n) = n + 1010$$

3.3 A Non-Trivial Solution

$$f(n) = \begin{cases} n+5 & n \text{ even} \\ n+2015 & n \text{ odd} \end{cases}$$

The choice of 5 and 2015 is arbitrary; any two odd numbers adding up to 2020 suffice.

Have we captured all the possibilities?

3.4 Manipulating the Function

With two alternative interpretations of $f(f(f(n)))$ we can deduce from 4 that:

$$f(n+2020) = f\left[f(f(n)\right] = f(f\left[f(n)\right]) = f(n) + 2020 \tag{5}$$

3.5 Reduction mod 2020

To proceed, instead of working with all integers $\mathbb{Z}$ we work with

$$\mathbb{Z}_{2020} = \{0, 1, 2, \ldots 2018, 2019\}$$

that is, the set of possible remainders on division by 2020. We say two integers are congruent modulo 2020 if their difference is a multiple of 2020, or equivalently, if they have the same remainder on division by 2020.

In the light of equation 5, we can define a modified function:

$$\tilde{f} : \mathbb{Z}_{2020} \mapsto \mathbb{Z}_{2020}$$

For $a \in \mathbb{Z}_{2020}$ we pick any $\mathbb{Z} \ni n \equiv a \mod 2020$ and define:

$$\tilde{f}(a) = f(n) \mod 2020$$

Equation 5 ensures this is well-defined, in that we get the same $\tilde{f}(a)$ regardless of which n we choose.

3.6 Twin Remainders

Unlike the original function f, the modified function f is self-inverse. Equation 4 implies that:

$$\tilde{f}(\tilde{f}(a)) = a; a \in \mathbb{Z}_{2020}$$

We refer to the pair $(a, \tilde{f}(a))$ as a *twin pair*. The self-inverse property ensures that if a is a twin of b then b is a twin of a. It also rules out any number being twinned by more than one other number.

Finally, we must show that no number is a twin of itself. To see why, we suppose there is a with $f(a) \equiv a \mod 2020$. Then there is some $c \in \mathbb{Z}$ such that:

$$f(a) = a + 2020c$$

But then equation 5 implies that:

$$a + 2020 = f(f(a)) = f(a + 2020c) = f(a) + 2020c = a + 4040c$$

Thus $c = 1/2$, contradicting $c \in \mathbb{Z}$.

Having excluded the possibility of self-twins, we conclude that $\tilde{f}$ induces a partition of $\mathbb{Z}_{2020}$ into disjoint twin pairs. Conversely, for any such partition we can define a self-inverse function $\tilde{f}$ that maps any number onto its twin.

3.7 A Shift Function

If we know $\tilde{f}$ we cannot uniquely reconstruct f. We can only reconstruct $f(n) \mod 2020$. In other words, if (a, b) is a twin, we do not know that $b = f(a)$. All we know is that for some shift function $s : \mathbb{Z}_{2020} \mapsto \mathbb{Z}$ we have:

$$f(n) = n + s(n \mod 2020)$$

Equation 5 ensures that $f(n) - n$ is periodic with period (a factor of) 2020.

We cannot choose the shift function arbitrarily. Firstly, for each twin pair (a, b) we must have $s(a) + s(b) = 2020$ in order to ensure equation 4 holds.

The shift function must also respect the twinning, so that for each twin pair (a, b):

$$s(a) \equiv b - a \mod 2020$$

This, together with $s(a) + s(b) = 2020$ implies the reverse relation:

$$s(b) \equiv a - b \mod 2020$$

3.8 General Solution

We are now in a position to describe the general solution to equation 4 as follows. The following algorithm generates all cases.

- Partition the set $\mathbb{Z}_{2020}$ into 1010 disjoint twin pairs, of the form (a, b).
- Pick a shift function $s : \mathbb{Z}_{2020} \mapsto \mathbb{Z}$ such that, for each twin (a, b):

$$s(a) + s(b) = 2020$$
$$s(a) \equiv b - a \mod 2020$$

- Define a function $f : \mathbb{Z} \mapsto \mathbb{Z}$ by:

$$f(n) = n + s(n \mod 2020)$$

- This function f satisfies equation 4, and ever solution is of this form.

3.9 It would not work if 2020 were odd

The solution relies on partitioning Z_{2020} into twin pairs. The same approach works if we replace 2020 with any even number.

It doesn't work if we replace 2020 with an odd number. In that case, there would be no functions f satisfying equation 4 .

4 Collatz Stopping Length

Functional equations might have a unique solution, multiple solutions or no solutions. And sometimes we do not know if a solution exists or not.

Let us consider functions from the positive integers to the integers: $f : \mathbb{Z}^+ \backslash \{0\} \mapsto \mathbb{Z}$

$$f(n) = \begin{cases} 0 & n = 1 \\ 1 + f\left(\frac{n}{2}\right) & n \text{ even} \\ 1 + f(3n + 1) & n \geq 3, n \text{ odd} \end{cases}$$

Is there a solution for $f(n)$? That depends on what happens to the map:

$$n \to \begin{cases} \frac{n}{2} & n \text{ even} \\ \frac{3}{n} + 1 & n \text{ odd} \end{cases}$$

- If the sequence ends up at 1, then $f(n)$ is the number of steps to get to 1.
- If the sequence shoots off to infinity, then we can define the first f to be whatever we like, and subtract 1 each step.
- If the sequence ends up in a cycle then there is no solution for f.

The *Collatz conjecture* is that all paths lead to 1 eventually.

5 Solution IMO 2019 Q1

The problem was to find all functions satisfying the equation 1.

The function of a function is awkward so we try to get rid of it by looking at two ways producing $f(f(n+1)$ on the right hand side, first as $0 + (n + 1)$ and then as $1 + n$

a	b	Equation 1
0	$n + 1$	$f(0) + 2f(n + 1) = f(f(n + 1))$
1	n	$f(2) + 2f(n) = f(f(n + 1))$

Equating the left hand sides of these two equations, we have:

$$f(n + 1) - f(n) = \frac{1}{2}(f(2) - f(0))$$

Let us call this quantity m, so f is linear and we have for all $a \in$.

$$f(a) = f(0) + ma$$

Substitute this into equation 1 and we have:

$$f(0) + 2ma + 2f(0) + 2mb = f(0) + m(f(0) + ma + mb)$$

Collecting all terms on the right hand side, we have:

$$\begin{aligned} 0 &= -3f(0) - 2m(a + b) + f(0) + mf(0) + m^2(a + b) \\ &= (m - 2)(m(a + b) + f(0)) \end{aligned}$$

Thus, either $m = 2$, in which case

$$f(a) = f(0) + 2a$$

for arbitrary $f(0)$. Or else, $m \neq 0$, in which case for all $a + b$,

$$m(a + b) = -f(0)$$

so $m = 0$ in which case $f(a) = 0$ for all a.