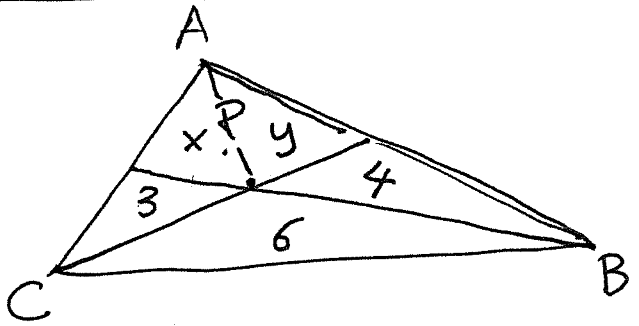
Mathematical Enrichment Feb 9th, 2019

Kevin Hutchison : Geometry

Some books:

Coxeter & Greitzer Geometry Revisited (Math. Assoc. of America M.A.A.)

Ross Honsbergers Mathematical Gems



$$\frac{x}{3} = \frac{x+y+4}{9}, \quad \frac{y}{4} = \frac{x+y+3}{10}$$

$$9x = 3x + 3y + 12 \qquad 10y = 4x + 4y + 12$$

$$6x - 3y = 12 \qquad 6y - 4x = 12$$

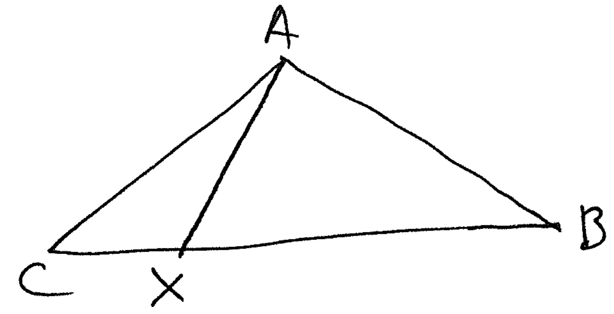
$$\boxed{2x - y = 4} \qquad \boxed{3y - 2x = 6}$$

add $\quad 2y = 10 \Rightarrow y = 5$

$$\Rightarrow x = \frac{9}{2}$$

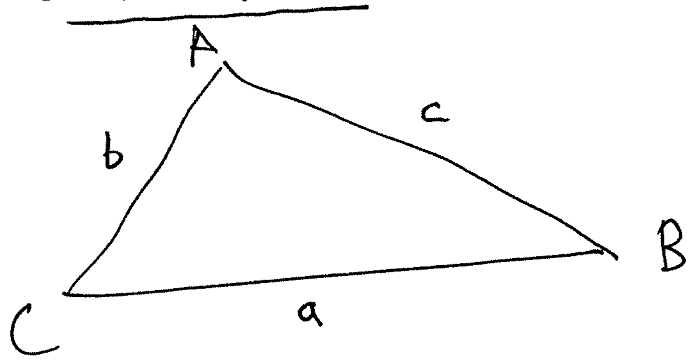
$$\therefore \; x + y = 5 + 9/2 = 19/2 = 9.5$$

<u>We used</u>



& $$\frac{\text{Area } ACX}{\text{Area } ABX} = \frac{CX}{XB}.$$

Sine Rule

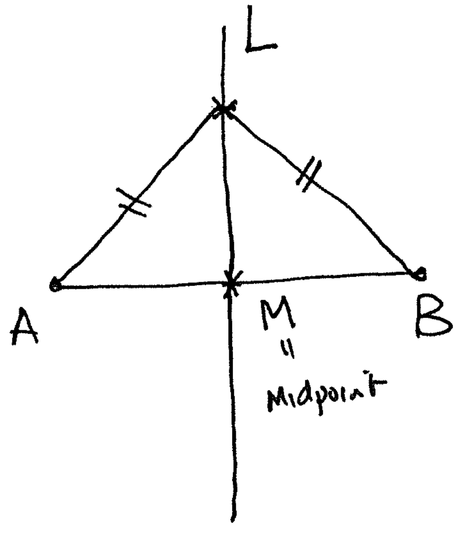


We saw $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = \; ?$

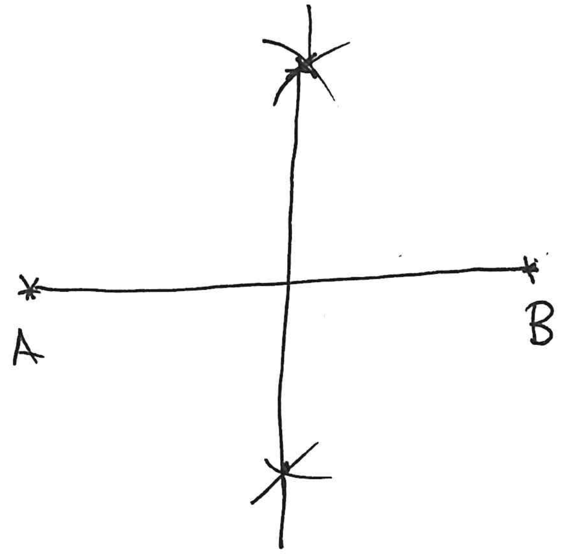
<u>Answer</u>: The diameter of the circumcircle of ABC

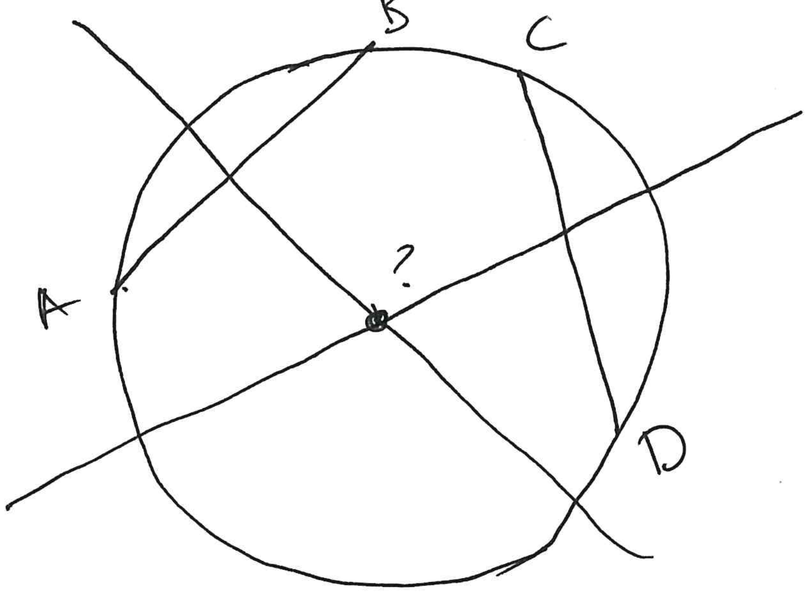
Circumcircle

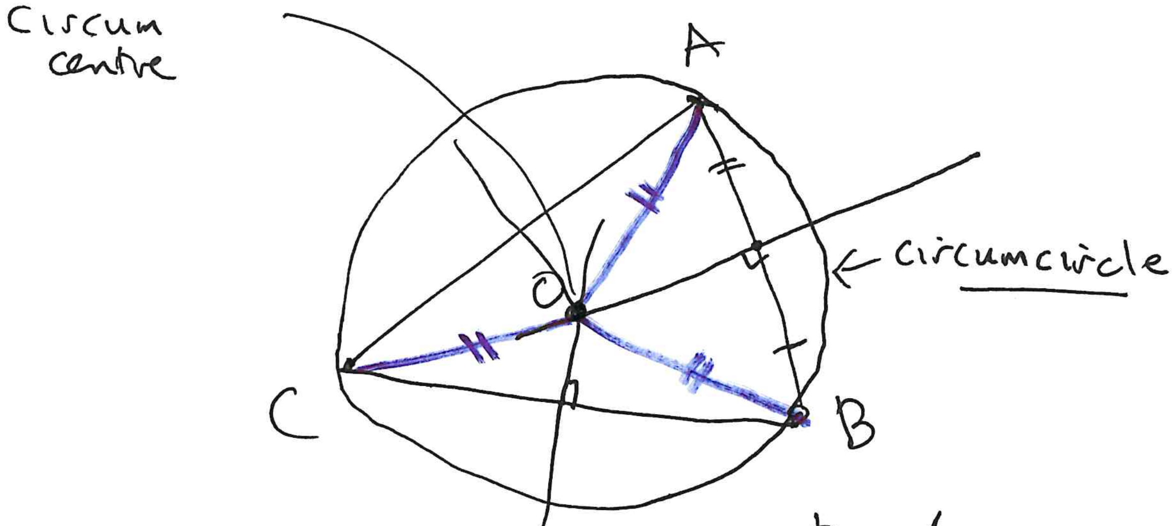
1. Perpendicular bisector of a line segment.



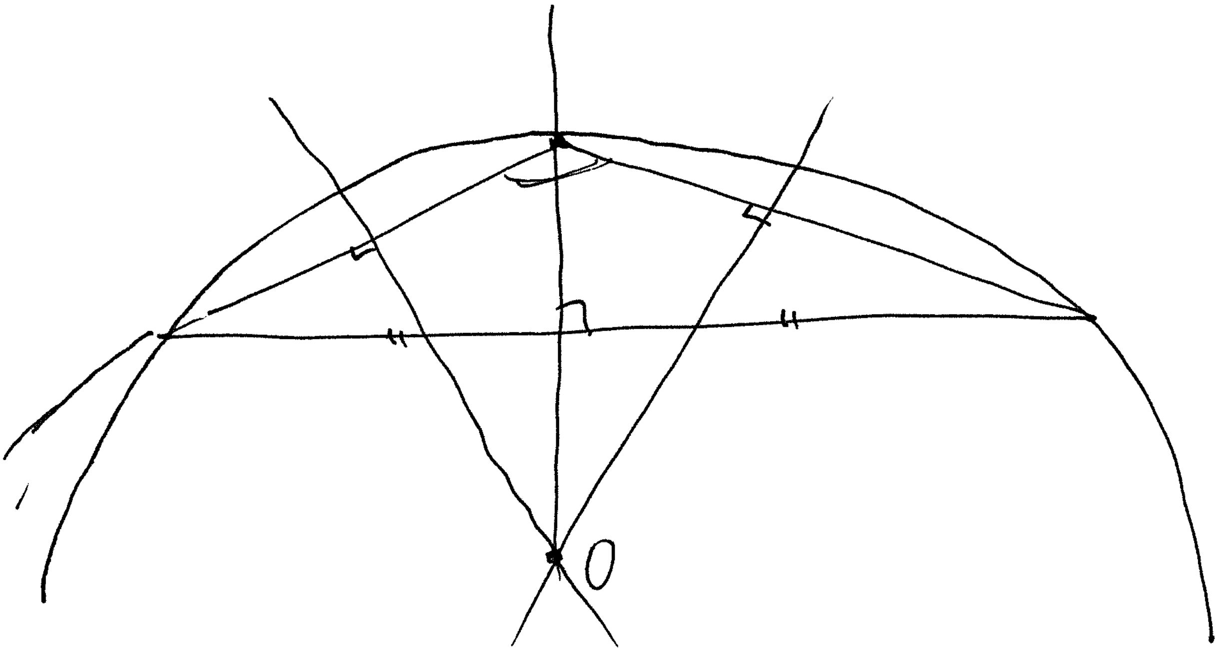
A point P in the plane lies on the perp. bisector L if and only if P is equidistant from A and B



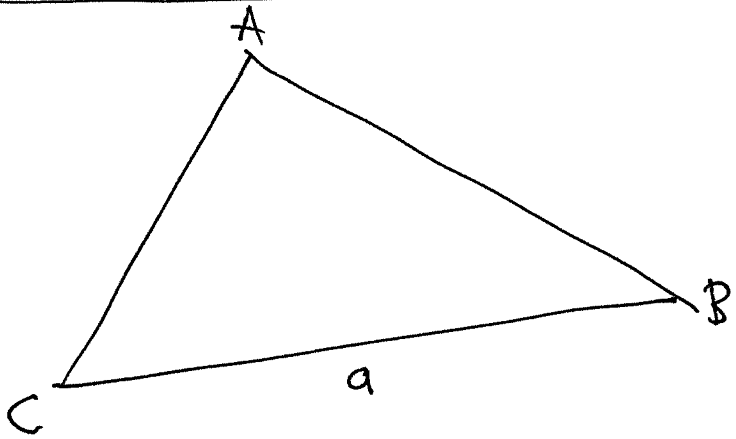




The circumcentre of a ~~circle~~ triangle is the point of intersection of the perp. bisectors of (any two) sides.

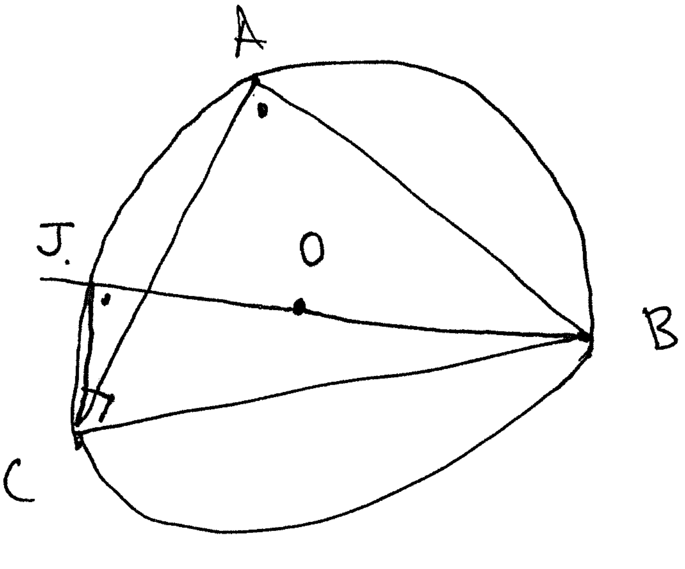


Circumcentre may lie outside Δ (if one angle is obtuse -- $> 90°$)



<u>Theorem</u> $\dfrac{a}{\sin A} = 2R$ where R = circumradius

<u>Proof:</u>



BO extended meets circle at J.

So $JB = 2R$

Also $\angle J = \angle A$ (angles standing on the same arc CB)(*)

$$\sin A = \sin J$$

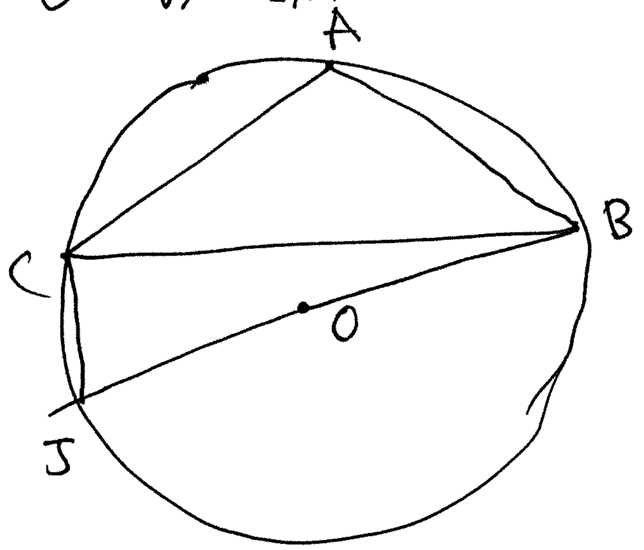
But $\angle JCB = 90^\circ$ (angle in a semicircle)(*)

So $\sin J = \dfrac{a}{JB} = \dfrac{a}{2R}$

$\therefore \sin A = \dfrac{a}{2R}$ <u>Done</u>

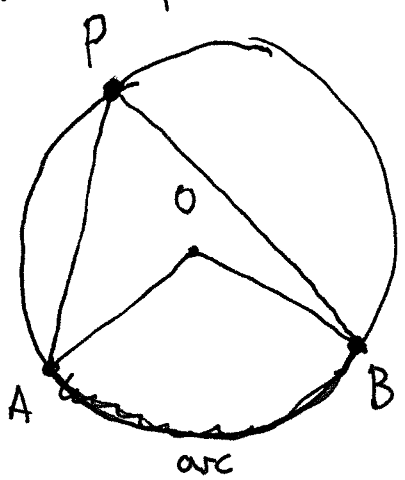
What is O is external? (*)

$\angle J = 180^\circ - \angle A$

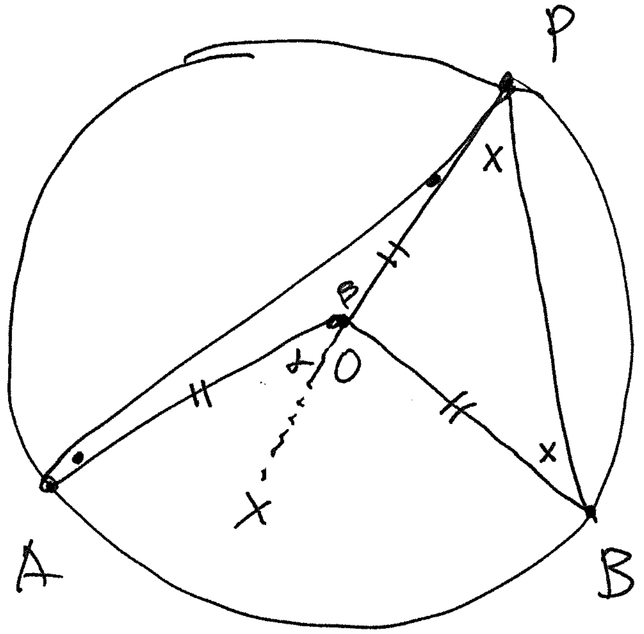


(*) <u>Theorem</u> The angle at a point on a circle standing on an arc is $\frac{1}{2}$ the angle at the centre of the circle standing on the same arc.

$\angle APB = \frac{1}{2} \angle AOB$



Proof:

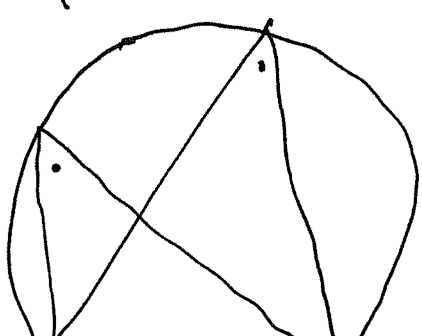


$$\angle AOX = 180^\circ - \angle AOP = \angle OAP + \angle APO$$
$$= 2\angle APO.$$

$$\angle BOX = 2\angle BPO$$

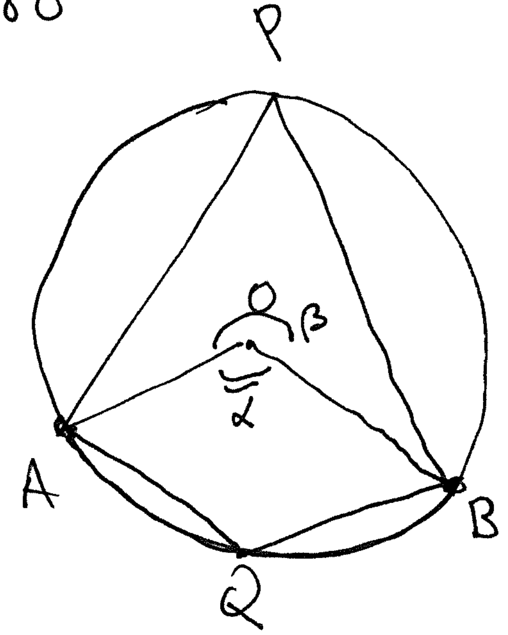
$\therefore$ Add. $\angle AOB = 2(\angle APO + \angle BPO)$
$$= 2\angle APB.$$

Corollary 1 All angles standing on the same arc are equal to each other



Corollary 2. ~~The~~ Any angle in a semicircle is a right angle. (Since AB is a diameter $\angle AOB = 180^\circ$)

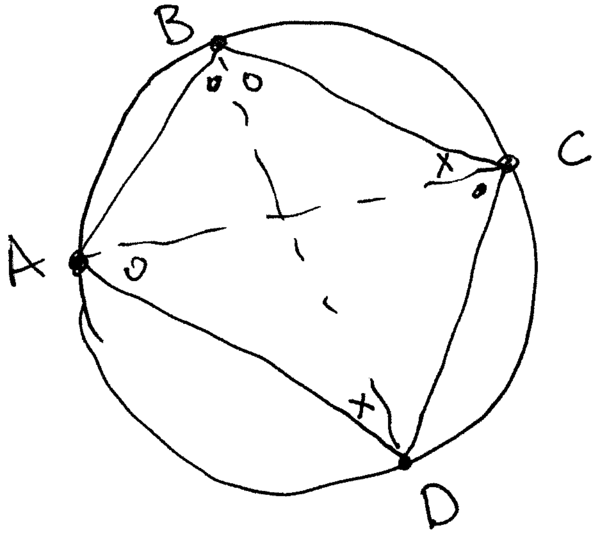
Corollary 3 Angles on complementary arcs sum to 180°



$$\angle P + \angle Q = 180^\circ$$

$$\left(\begin{aligned} \angle P &= \tfrac{1}{2}\alpha \\ \angle Q &= \tfrac{1}{2}\beta \\ \alpha + \beta &= 360^\circ \end{aligned} \right)$$

It follows that not every quadrilateral ABCD has a circumcircle.



If the quadrilateral ABCD can be inscribed in a circle -- we say it is a <u>cyclic</u> quadrilateral -- then

$$\angle B + \angle D = 180^\circ = \angle A + \angle C$$

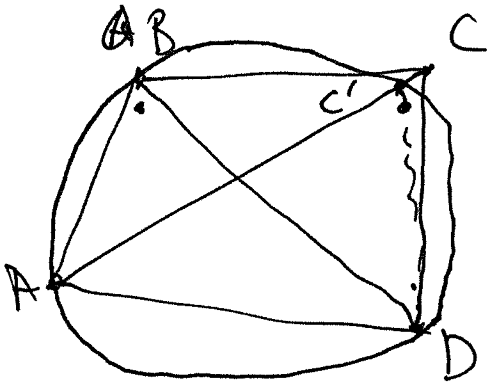
Also $\angle ABD = \angle ACD$ (standing on arc AD)

The converse statements are also true.

Let ABCD be any quadrilateral.

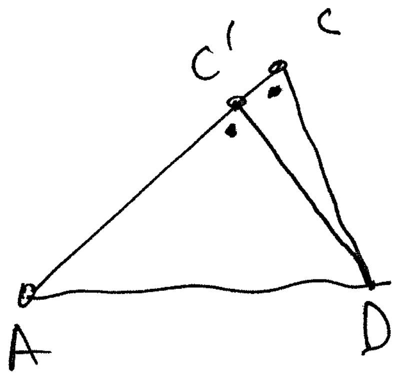
(1) If $\angle ABD = \angle ACD$ then ABCD is <u>cyclic</u>

<u>Proof:</u>



Take circumcircle of ABD. It meets AC at C', say.

$\angle AC'D = \angle ABD = \angle ACD$. ∴
(same arc)



This forces $C' = C$.

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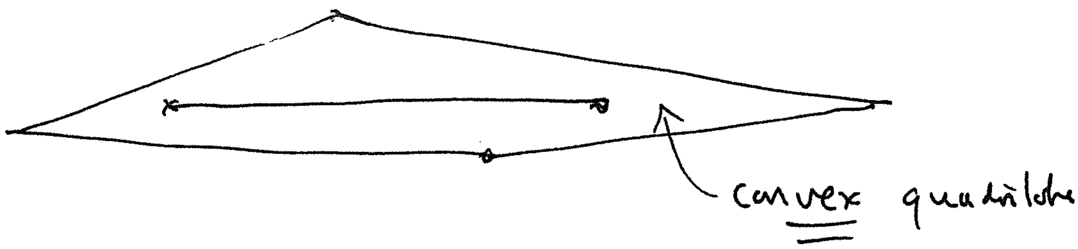
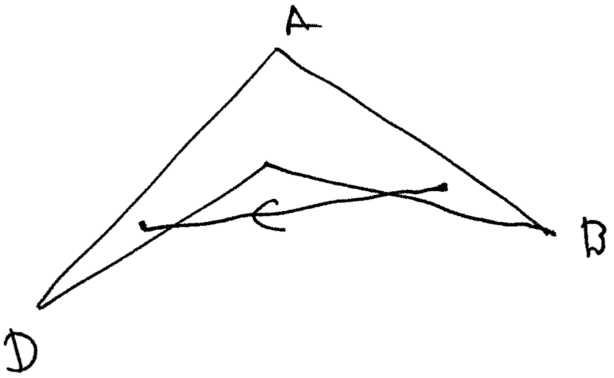
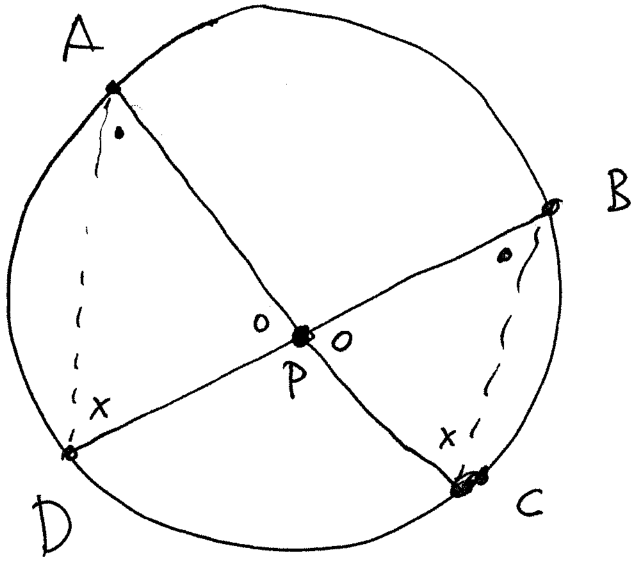
(2) If $\angle A + \angle C = 180°$ then ABCD is a <u>cyclic</u>. [Prove this!]

Most quadrilaterals are not cyclic.





not convex



Theorem $AP \cdot PC = DP \cdot PB$

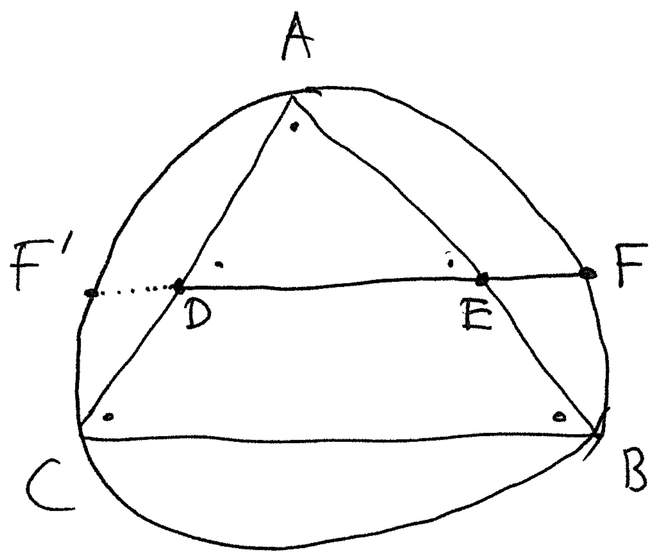
i.e. $\dfrac{AP}{DP} = \dfrac{BP}{PC}$

Proof: $\Delta DAP \sim \Delta CBP$ (same angles).

$\uparrow$ similar to

$\Rightarrow$ ratios $\dfrac{AP}{BP} = \dfrac{DP}{PC}$ (ratios of two corresponding sides are equal)

Question 2 from last time:



ABC equilateral.
D, E midpoints of AC, AB.

Prove

$$\frac{DF}{DE} = \frac{DE}{EF}$$

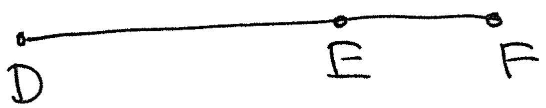
Solution ADE is also equilateral (all angles 60°)

Then consider A, F, B, F' :

$$\cancel{AE} \quad \frac{F'E}{AE} = \frac{EB}{EF}.$$

But $F'E = DF$ by symmetry

and $AE = DE. = EB.$ done

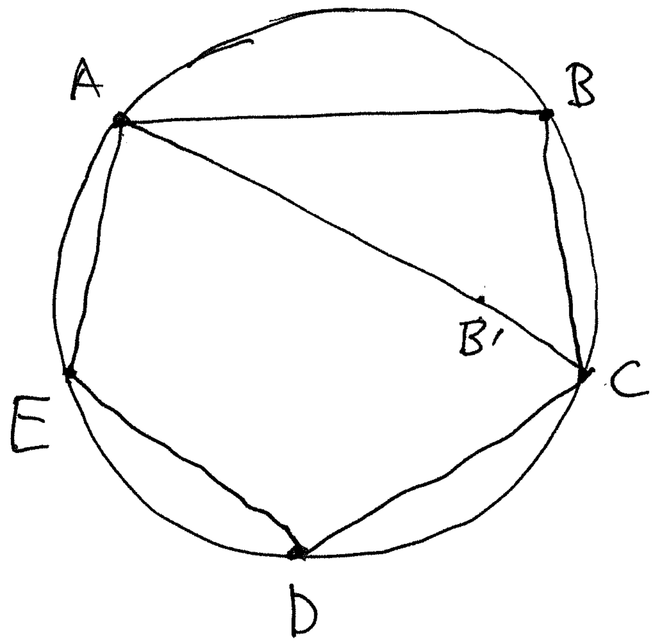


$$\frac{DF}{DE} = \frac{DE}{EF}$$

Exercise Show $\boxed{\dfrac{DF}{DE} = \dfrac{1+\sqrt{5}}{2}}$ golden ratio.

"Dividing segment in mean and extreme ratio"

Exercise A, B, C, D, E are equally distanced on a circle. "regular pentagon".



Show that

$$\frac{AC}{AB} = \frac{1+\sqrt{5}}{2}$$

Exercise Let ABC be an acute-angled triangle. Construct an equilateral triangle externally on each side.

Show that the circumcircles of these three triangles meet at a common point.