Mathematical Enrichment

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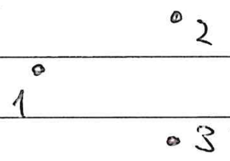
Sprouts (game)

[invented Feb 21, 1967 by J.H. Conway and M.S. Paterson at Cambridge]

- Game begins with n spots on a sheet of paper.

Example: $n=3$



- A move consists of drawing a line that joins one spot to another or to itself and then placing a new spot anywhere along the line

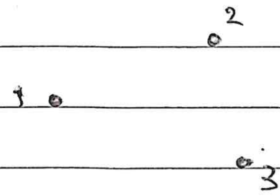
Restrictions: i) The line may have any shape but it must not cross itself/another line

ii) No spot may have more than three lines emanating from it.

- Two players. Players take turns drawing lines.
- The winner is last person able to draw a line.

Example $n=3$.

Start



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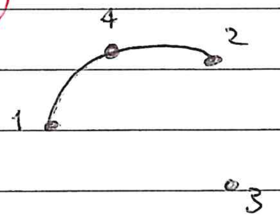

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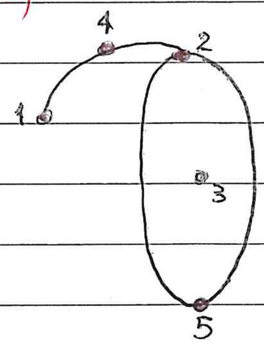

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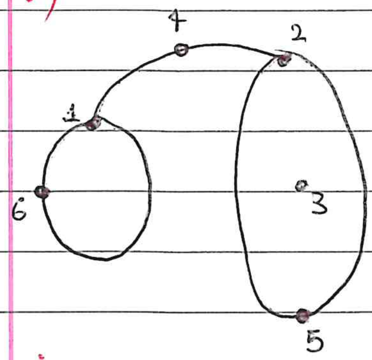

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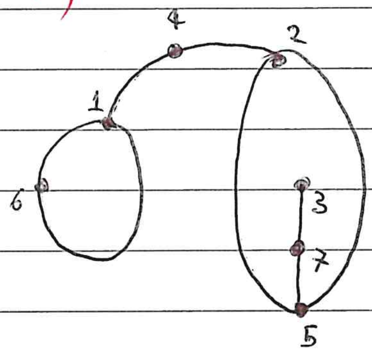

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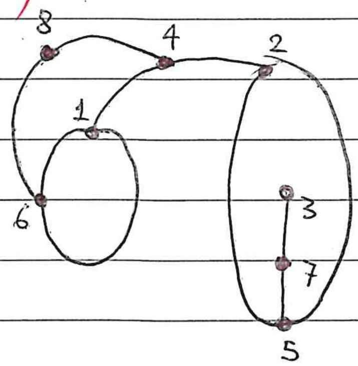

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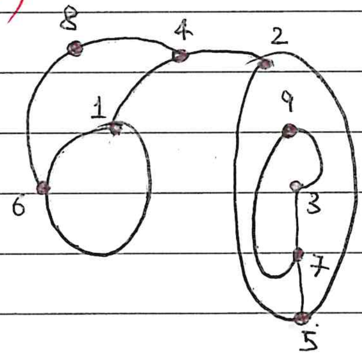

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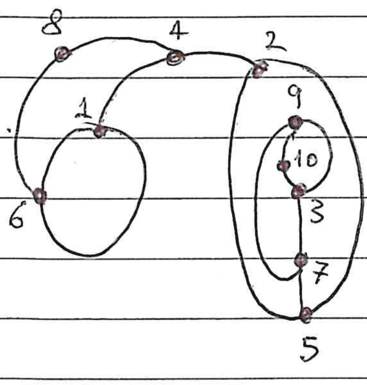


(A typical game of 3-spot sprouts)

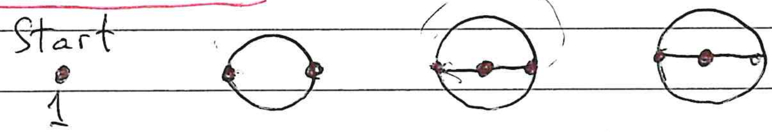
Start n spots → sprouts into fantastic patterns as game progresses.

Rmk: not merely a combinatorial game;

it exploits the topological properties of the plane

('Simple closed curves divide the plane into outside and inside regions')

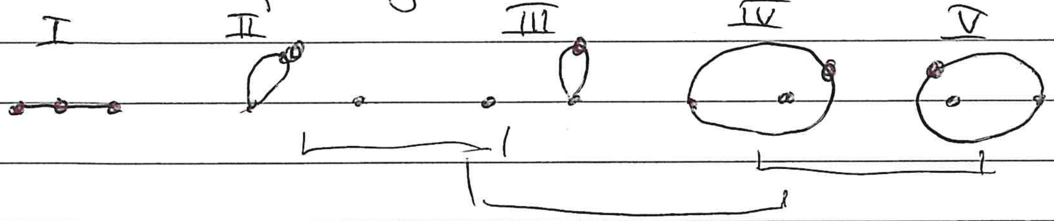
Analysis of the game:

1 spot : 

1^{st} player ~~ah~~ wins always.

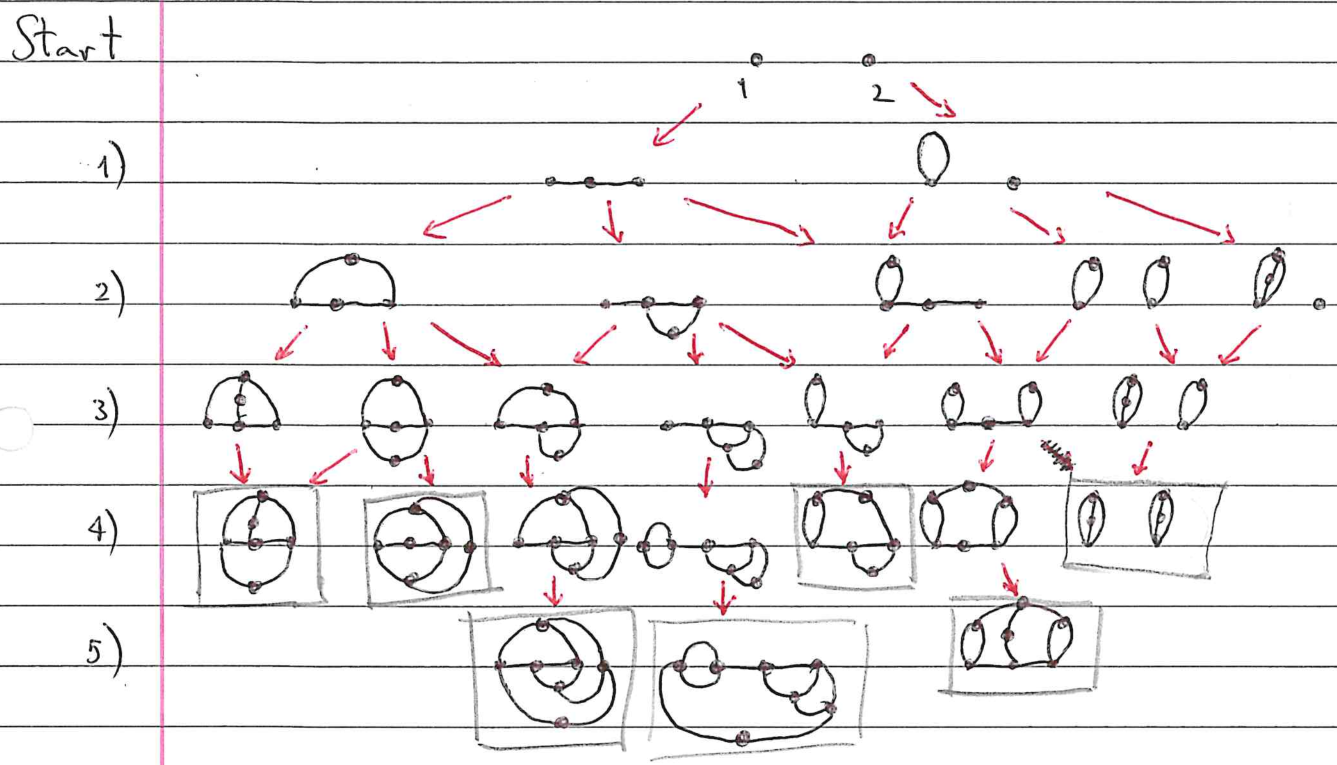
2 spots : Start $\cdot_1 \quad \cdot_2$

Possible opening moves:



II, III, IV, V are equivalent.
(think as playing on surface of sphere).

<u>Exercise</u>: draw chart tree for the 2-spot game and the 3-spot game.



Q: Can a sprout game keep sprouting forever?

<u>Thm</u> For ~~a~~ n spots the game must end in at most $3n-1$ moves.

<u>Proof</u>. Each spot has three 'lives'.

Each move kills two lives, at the beginning and at the end of the curve, but adds a new spot with one life. A game cannot continue when only one life remains. So, no game can last beyond $3n-1$ moves. $\square$

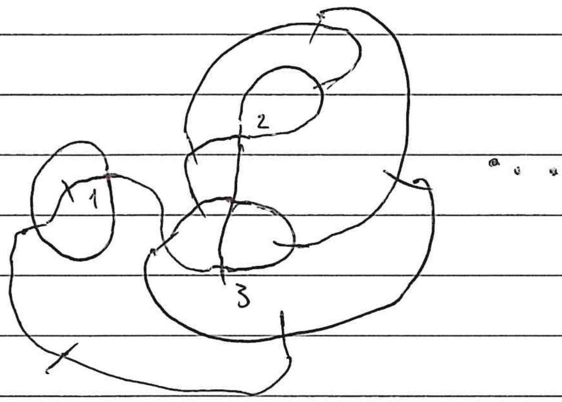
Thm Every game must last at least $2n$ moves.

Proof. Exercise.

Brussel sprouts

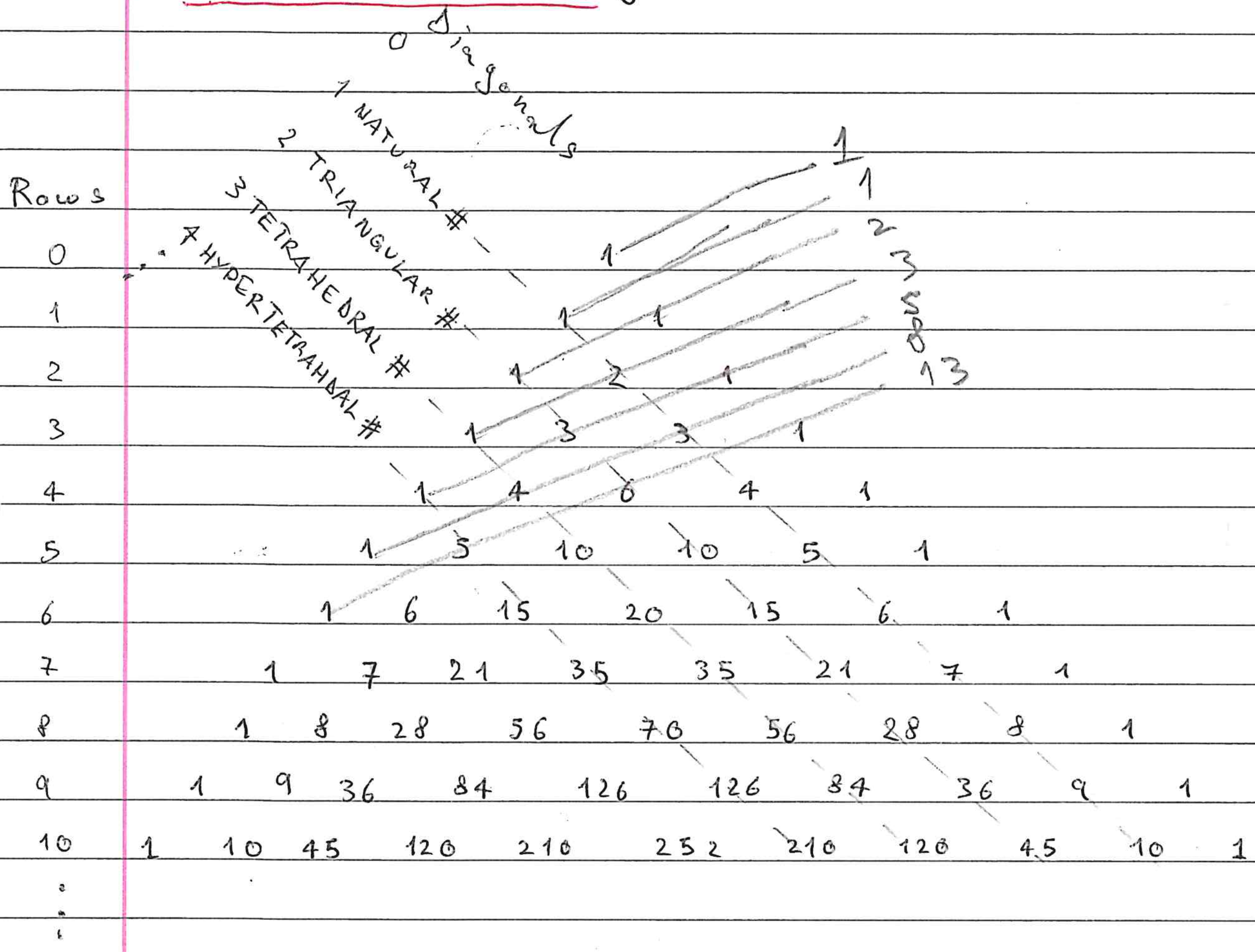
- Game begins with n crosses;
- A move consists of extending any arm of any cross into a curve that ends at the free arm of any other cross; then a crossbar is created along the curve to make a new cross. Same restrictions as in sprouts.

Example.



Thm Every Brussel sprouts game must end in exactly $5n-2$ moves!

Pascal's Triangle



One of the most famous patterns in the history of mathematics.

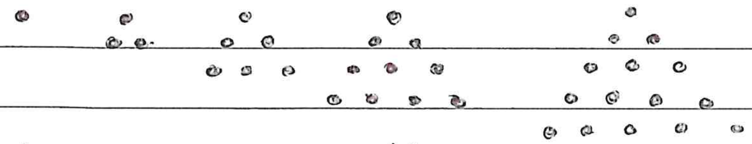
The triangle begin with 1 at the apex.

All other numbers are the sum of the two numbers directly above them.

Patterns in Pascal's triangle

- Diagonal rows: triangular numbers and analogues in space of all dimensions.

Examples: 2^{nd} diagonal $\to$ Δ-numbers in dim $= 2$



1 3 6 10 15

1^{st} diagonal $\to$ natural numbers (analogue of Δ-numbers in dim $= 1$)

0^{th} diagonal $\to$ dim $= 0$: the point itself is the only possible pattern

3^{rd} diagonal $\to$ tetrahedral numbers : cardinality of sets of points that form tetrahedral arrays in 3-dim space.

n^{th} diagonal $\to$ n-dim analogue of Δ-numbers.

Q: find the sum of all numbers in any diagonal, down to any place in the serie.

Example : • sum the natural numbers from 1 to 9.

• what is the sum of the first eighth Δ-numbers?

- More gently sloping diagonals: their sum form a well-known sequence

$$1, 1, 2, 3, 5, 8, 13, \ldots \qquad \begin{cases} F_0 = 1, \; F_1 = 1 \\ F_k = F_{k-1} + F_{k-2}, \; k \geq \end{cases}$$

Can you see why?

- Horizontal rows: coefficients in the expansion of the binomial $(x+y)^n$.

Example: $(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$

Binomial theorem: $(x+y)^n = \sum_{k=0}^{n} \underbrace{\binom{n}{k}}_{\text{binomial coeff.}} x^k y^{n-k}$

Basic tool in elementary combinatorics and probability.

Toss three coins: eight possible outcomes

HHH, HHT, HTH, HTT, THH, THT, TTH, TTT.

One way to 3 H's, three to get 2 H's, 3 to get 1'H one way to get 0 H's.

1 3 3 1 $\leftarrow$ 3^{rd} raw of the triangle.

Ex. Toss a coin 9 times.
What is the probability of getting exacty 4 heads?

- \# ways of selecting four coins from ~~ten~~ nine.

= intersection of diagonal 4 and row 9

= 126 $\left(\text{this is } \binom{9}{4} = \frac{9!}{5!4!} \right)$

- \# all possible outcomes

= sum of the numbers in the 9^{th} row.

Fact: sum of n^{th} row of triangle $= 2^n$

(Why? Use Binomial theorem.

Alternatively, observe that sum of each row

is twice the sum of the preceding row...).

So, P(getting exacty 4 heads)

$$= \frac{126}{2^9} = \frac{126}{512} = \frac{63}{256} \quad (\approx 24.6\%)$$

Further questions:

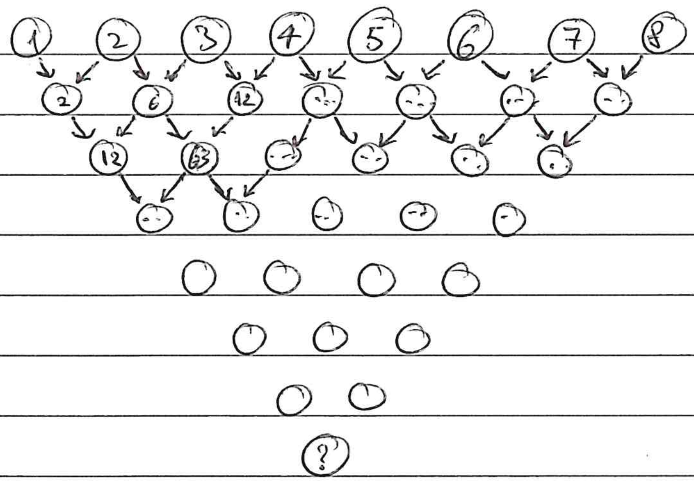
1. What formula gives the ~~number~~ sum of all numbers above row n?

2. How many odd numbers are there in row 255?

3. How many numbers in row 67 are divisible by 67?

4. Given an initial row of n card, how can one obtain from Pascal's triangle simple formulae for calculating the apex in the pyramid trick?

5. Draw a triangular scheme as below



Multiply the numbers according to the rule above. With how many zeros will end the number in the last circle at the bottom?

Answers:

1) $1 + 2 + 2^2 + \dots + 2^{n-1} = 2^n - 1$

2) Let $f(n) = \#$ odd values in the n^{th} row.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$f(n)$	1	2	2	4	2	4	4	8	2	4	4	8	4	8	8	16	2

Notice the pattern:

3rd, 7th, 15th

1 | 2 | 2, 4 | 2, 4, 4, 8 | 2, 4, 4, 8, 4, 8, 8, 16 | 2, 4, 4, 8, 4, ...

1st row, 2nd, 4th, 8th, 16th

We conjecture that: if $n = 2^k$ $(k \geq 1)$, then the n^{th} row contains only even numbers, except for the two 1's on the ends;

if $n = 2^k - 1$, then the n^{th} row contains only odd numbers.

Answer: Row 255: all numbers are odd.

$(255 = 256 - 1 = 2^8 - 1)$

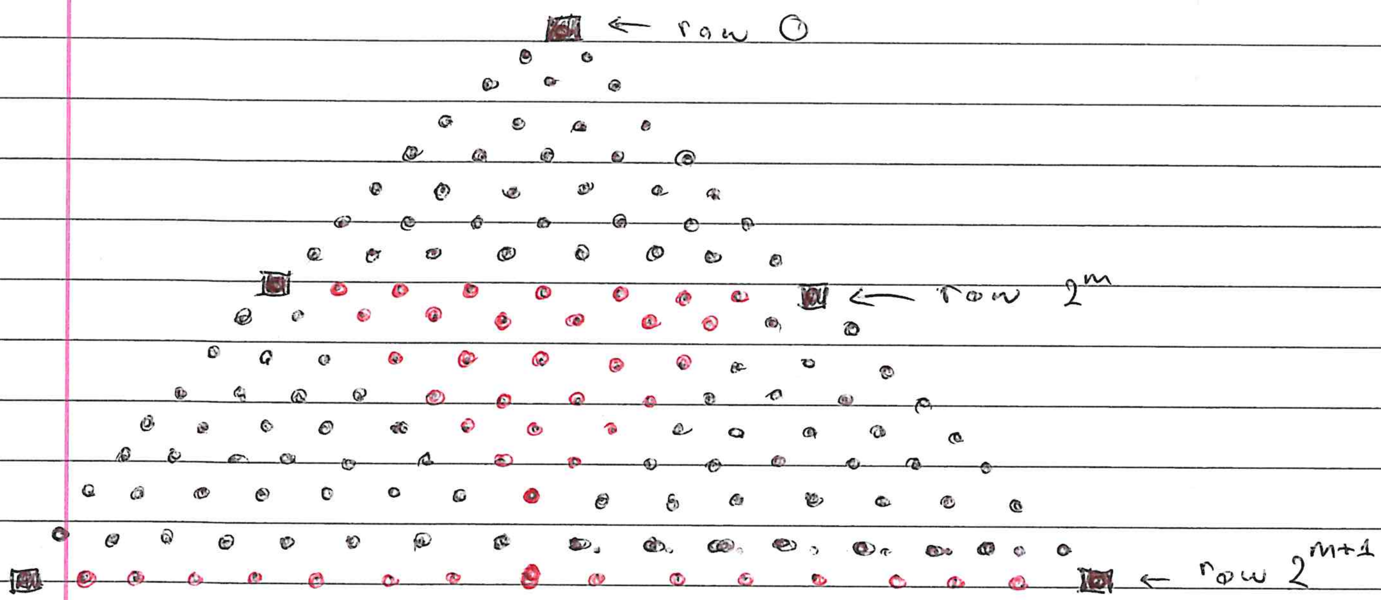
Thm Row 2^m contains only even numbers, except for the two 1's at the ends of the row.

Proof by induction:

- Base case: $m=1 \to 2^1 = 2$

2^{nd} row is $1, 2, 1$. True.

- Induction step: wts true for $2^m \Rightarrow$ true for 2^{m+1}.



Corollary Row $2^m - 1$ contains only odd numbers.

The birthday problem

30 people selected at random.

What is the probability that at least two of them have the same birthday (that is, same month and day)?

(a) ~ 1% (b) ~ 5% (c) ~ 20% (d) ~ 70%.

'Easier' to compute probability that all birthdays are different.

Prob. any two people have different bday $= \frac{364}{365}$.

Prob. a third person's bday differs from the other two $= \frac{363}{365}$.

a fourth person's, $\frac{362}{365}$, ..., the 30^{th} person $\frac{336}{365}$.

So, probability that no one share bday is

$$\frac{365 \cdot 364 \cdot 363 \cdots 336}{365 \cdot 365 \cdot 365 \cdots 365} \simeq 30\%$$

$\Rightarrow$ Prob. at least two share bday is $\simeq 70\%$!

Derangements

Two people, each hold a shuffled deck of 52 cards. They count aloud from 1 to 52; on each count both deal a card face up on the table. What is the probability that at some point two identical cards will be dealt simultaneously?

(The same problem is met in other forms. 'A distracted secretary puts a number of letters at random into addressed envelopes. What is the probability of at least one letter reaching the right person?').